EIGHTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The eighth annual meeting of the Mathematical Association of America was held at the University of Cincinnati on Thursday and Friday, December 27 and 28, 1923, in affiliation with the American Association for the Advancement of Science, and in conjunction with the western meeting of the American Mathematical Society. There were 158 present at the various sessions, including the following 108 members of the Association:

J. W. ALEXANDER, Princeton University.
R. B. ALLEN, Kenyon College.
W. E. ANDERSON, Miami University.
R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.
C. S. ATCHISON, Washington & Jefferson College.
A. C. BAIRD, Fifth Ave. High School, Pittsburgh.
J. W. BALDWIN, College of City of Detroit.
GRACE M. BAREIS, Ohio State University.
I. A. BARNETT, University of Cincinnati.
L. A. BAUER, Dept. of Terrestrial Magnetism.
E. M. BERRY, Purdue University.
HAROLD BLAIR, Western (Mich.) State Normal College.
R. L. BORGER, Ohio University.
P. P. BOYD, University of Kentucky.
J. W. BRADSHAW, University of Michigan.
LOUIS BRAND, University of Cincinnati.
W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, University of California.
A. D. CAMPBELL, University of Arkansas.
C. J. CARDIN, Washington & Jefferson College.
R. D. CARMICHAEL, University of Illinois.
A. B. COBLE, University of Illinois.
H. H. CONWELL, Beloit College.
N. A. COURT, University of Oklahoma.
S. E. CROWE, Michigan Agricultural College.
H. T. DAVIS, Indiana University.
S. C. DAVISSON, Indiana University.
W. W. DENTON, University of Michigan.
L. E. DICKSON, University of Chicago.
C. S. DOAN, Purdue University.
B. F. DOSTAL, University of Michigan.
H. H. DOWNING, University of Kentucky.
ARNOLD DRESDEN, University of Wisconsin.
W. E. EDINGTON, Purdue University.
L. C. EMMONS, Michigan Agricultural College.
B. F. FINKEL, Drury College.
T. M. FOCKE, Case School of Applied Science.
TOMLINSON FORT, University of Alabama.
A. F. FRUMVELLER, Marquette University.
O. T. GECKELER, Carnegie Institute of Technology.
F. J. GERST, St. Xavier College.
E. D. GRANT, Earlham College.
V. G. GROVE, Michigan Agricultural College.
J. M. GUILLIAMS, Berea College.
HARRIS HANCOCK, University of Cincinnati.
JULIA L. HAWKINS, Oklahoma College for Women.
C. T. HAZARD, Purdue University.
E. R. HEDRICK, University of Missouri.
E. J. HIRSCHLER, Bluffton College.
DUNHAM JACKSON, University of Minnesota.
H. M. JEFFERS, University of Iowa.
L. C. KARPINSKI, University of Michigan.
J. H. KINDLE, University of Cincinnati.
G. A. KNAPP, Maryville College.
H. W. KUHN, Ohio State University.
SOLOMON LEFSCHETZ, University of Kansas.
H. B. LEMON, Denison University.
F. ELIZABETH LE STOURGEON, University of Kentucky.
ANNA D. LEWIS, Lake Erie College.
S. L. MACDONALD, Colorado Agricultural College.
W. D. MACMILLAN, University of Chicago.
E. S. MANSON, Jr., Ohio State University.
WILLIAM MARSHALL, Purdue University.
T. E. MASON, Purdue University.
G. A. MILLER, University of Illinois.
C. N. MOORE, University of Cincinnati.
R. L. MOORE, University of Texas.
C. C. MORRIS, Ohio State University.
F. R. MOULTON, University of Chicago.
PAUL MUEHLMAN, Loyola University.
I. C. NICHOLS, Louisiana State University.
H. L. OLSON, University of Michigan.
J. R. OVERMAN, Bowling Green (Ohio) State Normal College.
ANNA H. PALMIÉ, Western Reserve College for Women.
L. C. PLANT, Michigan Agricultural College.
S. E. RASOR, Ohio State University.
C. N. REYNOLDS, Jr., West Virginia University.

H. L. RIETZ, University of Iowa.
E. D. ROE, Jr., Syracuse University.
W. H. ROEVER, Washington University.
D. A. ROTHROCK, Indiana University.
H. E. RUSSELL, University of Denver.
F. H. SAFFORD, University of Pennsylvania.
IDA M. SCHOTTENFELS, Chicago, Ill.
E. W. SCHREIBER, Proviso Twp. High School, Maywood, Ill.
W. G. SIMON, Western Reserve University.
MARY E. SINCLAIR, Oberlin College.
H. E. SLAUGHT, University of Chicago.
E. R. SMITH, Iowa State College.
E. S. SMITH, University of Cincinnati.
I. W. SMITH, N. D. Agricultural College.
C. P. SOUSLEY, Rose Polytechnic Institute.
G. W. SPENCELEY, Miami University.
W. M. STEIRNAGLE, Sturgis, Mich.
R. B. STONE, Purdue University.
K. D. SWARTZEL, University of Pittsburgh.
T. E. TROTT, Mount Union College.
BIRD M. TURNER, West Virginia University.
H. S. VANDIVER, Cornell University.
E. B. VAN VLECK, University of Wisconsin.
OSWALD VEBLEN, Princeton University.
J. H. WEAVER, Ohio State University.
B. F. YANNEY, College of Wooster.
E. I. YOWELL, University of Cincinnati.
J. W. YOUNG, Dartmouth College.
W. A. ZEHRING, Purdue University.

Dean P. C. NASH, Antioch College, Official Representative.

The opening meeting of the American Association was held on Thursday evening in Emery Auditorium, presided over by Professor L. T. More, chairman of the local committee. A welcome was extended by the mayor and by the president of the University of Cincinnati, after which President C. D. Walcott introduced the retiring president, Professor J. Playfair McMurrich. His address "A Retrospect" appeared in the issue of *Science* for December 28, 1923. Among other meetings on the general program were the Sigma Xi lecture on "The vacuum, There's something in it" under the joint auspices of the Association and the Society of Sigma Xi on Friday evening, by Dr. W. R. Whitney, director of the Research Laboratory of the General Electric Company; the commemorative meeting on Saturday evening devoted to the history of the Association; and a complimentary concert of Shakespearean music given on Sunday afternoon to the visiting guests through the courtesy of the Cincinnati Orchestra Association.

At the meeting of the American Association Council on Saturday morning Dr. J. McK. Cattell was elected president for the coming year. Professor J. C. Fields of the University of Toronto was elected chairman of Section A, Professor J. A. Miller of Swarthmore College chairman of Section D, and Professor L. C. Karpinski of the University of Michigan chairman of Section L. Professor G. A. Miller of the University of Illinois was elected a member of the Executive Committee of the American Association, on nomination of the mathematicians present, to fill the vacancy caused by the election of Dr. Cattell as president. Professor Fields was appointed delegate from the American Association on the American Section of the International Mathematical Union with Professor G. A. Miller as alternate.

A great deal of interest was aroused through the special prize of $1000.00 offered by a member of the American Association for a notable contribution to the advancement of science made either before one of the sections of the Association or before one of the societies that met with the Association at Cincinnati. The address by Professor L. E. Dickson on "Algebras and their arithmetics" was of so outstanding a character that the mathematics secretaries recommended this strongly to the Committee on Award, and at the close of the Cincinnati

meetings this award was made to Professor Dickson. An abstract of this paper appears later in this report.

The joint dinner of mathematicians and astronomers was held on Friday evening at the Fountain Room in Hotel Gibson. Following the dinner Dr. J. A. Shohat, formerly of Petrograd, now assisting at the University of Chicago, expressed the appreciation of the Russian scientists and mathematicians for the aid which the United States has extended in saving the Russians from starvation. Professor Slaught explained the character of the By-Laws of the newly incorporated American Mathematical Society, and the relation thereto of the former Chicago Section; there are now three vice-presidents, one of whom will be the presiding officer for the western meetings of the Society, and an additional secretary serving in the place of the former secretary of the Section. Professor Hedrick explained that the tremendous increase in the cost of printing necessitated the printing of the *Bulletin* and the *Transactions* in Hamburg for the present, saying that in the case of the *Bulletin* the cost, which was $2000.00 before the war and $2800.00 about 1920, would be about $4800.00 on the present schedule, and that this would be well beyond the means of the Society to carry. Professor Dresden reported on the great need of completing the proposed endowment fund of $100,000.00 for the Society and stated that the subscriptions from members had amounted to nearly $25,000.00. Whether the subscription rate of one's salary for one day for each of the five years, as suggested by the Endowment Committee, be followed or not, the appeal was made to the members of the Society to look upon the matter not as charity but as a call to take a loyal part in this work even if only a small part. Professor Veblen, the president of the Society, spoke of the National Research Council Fellowships which the Rockefeller Foundation has recently voted to extend through a second five-year period, whereby there will be a number of fellowships in mathematics, physics, and chemistry to be divided among these three departments in accordance with the merits of the candidates. He said that it is now very important to make this plan a success and that this depends on the mathematicians of the country. The appointments are made for one year but are usually continued for three years or even for four years. The amount for the first year is $1800.00, but it is stipulated that it shall start at $2300.00 if the candidate has dependents. The appointments are ordinarily made to those who have already attained the degree of Doctor of Philosophy, and the liberal compensation is intended to compete with instructorships in colleges or universities. The suggestion was made that applications be sent at once without waiting for the formal announcement of the plan. Aside from various informal remarks in a lighter vein by various members, Professor Slaught voiced the sentiments of all present in recognition of the able and successful efforts of President Veblen in securing the extension of the above-mentioned plan so as to include mathematicians in its scope, and for his effective service in behalf of mathematics in his capacity as chairman of the Physical Science Division of the National Research Council.

Professor C. N. Moore acted as chairman of the committee on arrangements

with Professors Louis Brand and J. H. Kindle as his associates. Professor Moore was also very active and effective as the chairman of the committee on publicity for the general Association purposes. A vote of appreciation of the hospitality shown by Professor Harris Hancock and his colleagues of the department of mathematics at the University of Cincinnati, and of the courtesies and convenient arrangements made by the University of Cincinnati for these meetings, was offered at the joint session of the mathematical organizations by Professor Dunham Jackson, and was heartily adopted.

President Carmichael was introduced for his retiring address by ex-president Slaught, and President Carmichael then occupied the chair until after the election on Friday morning at which time he called upon President Rietz to preside.

The program committee consisted of Professor H. L. Rietz, Chairman, Dr. Mayme I. Logsdon, and Professor C. C. Morris. The following program was given, abstracts of these papers being given with numbers corresponding to the numbers in the list of titles. Professor T. H. Hildebrandt was unable to give his paper on "Outlines of fields of research: General analysis" because of illness in his family.

FIRST SESSION OF THE ASSOCIATION.

(1) "The present state of the difference calculus and the prospect for the future" by Professor R. D. CARMICHAEL, University of Illinois, retiring president of the Association.

(2) "Modern synthetic geometry and the college curriculum" by Professor N. A. COURT, University of Oklahoma.

Discussion opened by Professor J. H. WEAVER, Ohio State University.

(3) "The nature and function of descriptive geometry" by Professor W. H. ROEVER, Washington University.

Discussion opened by Professor J. W. BRADSHAW, University of Michigan.

1. President Carmichael's address appears in full in this number of the MONTHLY.

2. Modern geometry as presented by Professor Court is a direct continuation of elementary geometry, and in his judgment no college student interested in mathematics should ignore such notions as Ceva's theorem, the nine-point circle, the Simson line, coaxal circles, etc. The lack in the curriculum of such a course leads the student, by comparison with the classical course in college algebra, to the erroneous belief that the subject of plane Euclidean geometry was exhausted by his high school text-book. For teachers and prospective teachers of high school geometry the study of modern geometry is very essential. The traditional college course in mathematics followed by prospective high school teachers does not supplement or extend the student's knowledge of Euclidean geometry, and does nothing whatever to increase and develop the student's skill in handling geometric material by those methods which the prospective teacher will have to use in his work with high school students of plane geometry. Professor Court suggested the subject matter of such a course, saying that the

material is abundant, the methods are simple, and the work requires a minimum of preliminary training. It is his hope, and as well his experience, that modern geometry when once introduced will attract a large number of enthusiastic workers from among the secondary school teachers of mathematics. This course is preëminently adapted to our summer schools which are becoming more and more popular among high school teachers in active service.

Professor Court's plan will be presented in full in the May MONTHLY.

Professor Weaver emphasized the arguments in favor of modern geometry as a college study advanced by Professor Court and added that this affords a good foundation course for the student of advanced mathematics, in that it introduces him in a concrete way to a knowledge of fundamental notions which will be useful in various branches of advanced mathematics, and affords a chance to do a certain type of logical thinking which can be obtained nowhere else quite so readily. He pointed out that other countries than the United States actually require such preparatory courses (see Bulletin, 1917, No. 27, U. S. Bureau of Education, "The Training of Teachers of Mathematics" by Professor R. C. Archibald).

Professor Slaught's inquiry brought out the fact that nineteen institutions represented at the meeting have offered such a course in recent years. He spoke too of the care with which Professor Court has gone into the subject and the very concrete way in which he has prepared a definite and well-tried course which is at present in mimeographed form. He urged that in any feasible fashion the members of the Association might well encourage the preparation of text-books for such a course.

In view of the apparent interest manifested in this question, an informal vote of those present on the desirability of having a suitable text-book for courses in College Geometry was unanimously passed. Various suggestions were made as to the subject matter of such a text.

Professor Archibald told of a semester course of this sort at Brown University which uses as a text Petersen's "*Methods and theories for the solution of geometric problems of construction*," an English translation of which is now readily available. In a second semester he uses Hadamard's geometry with its important appendices containing the notions of area, of volume, etc., also Klein's "*Famous problems of antiquity*" for five weeks and Young's "*Fundamental concepts*,"—a very full and rich year, interlarded as it is with many matters of history connected with these questions.

3. In the paper by Professor Roever the principal purposes of the subject were defined to be (1) representation of the objects of space by means of figures which lie in a plane (or upon a surface), and (2) solution of the problems of space by means of constructions which can be executed in the plane. It was shown that the requirement for a picture to be adequate, *i.e.*, capable of producing a retinal image differing but little from that produced by the object itself, naturally leads to the use of central or parallel (orthographic or oblique) projection as a means of representation; and also that the requirement for an unambiguous correspondence

between space and the plane results in the need for two projections or for one projection with information concerning the object (such, for instance, as the perpendicularity of edges).

The problems of space were classified and the solution of fundamental ones was illustrated by means of the principal methods of descriptive geometry, namely, by the Mongean method, the axonometric method, the topographic method, and the method of free perspective. Among the applications considered were: the solution of spherical triangles and the problem of dialing; mitre joints for both straight and curved mouldings; intersection of arches; construction of lines of force for fields of force composed of combinations of simpler fields, for logarithmic potential and distributions of revolution in Newtonian potential; intersection of surfaces represented by the topographic method and by other methods; determination of the boundary curve of the projection of a surface.

Finally, need for the production of good pictures in books on mathematics was stressed, and the value of the study of descriptive geometry as a means of developing the power of space visualization was emphasized. The reader interested in a brief account of the subject is referred to the author's paper entitled "Descriptive geometry and its merits as a collegiate as well as an engineering subject" published in the MONTHLY (*1918*, 145–159).

Professor Bradshaw spoke of a recent book on calculus in which is found a so-called "Standard figure of the ellipsoid," a critical examination of which from the standpoint of descriptive geometry reveals three different directions of projection. In contrast with this he mentioned another recent book in calculus which contains the familiar problem of a tree cut halfway through by a horizontal section and a section making 45 degrees with the horizontal; a half-page is devoted to a thoroughly accurate drawing together with clear and correct directions to the student for drawing the figure, indicating, for example, that the tangents to the two curves (ellipses) which are parallel to the diameter of the tree should appear so in the drawing.

An inquiry from Professor Macdonald brought out the fact that in the case of Washington University and of the University of Michigan, descriptive geometry is given as a course in pure mathematics; a course at Michigan being given in the literary college distinct from the engineering course, and being a lecture course for prospective teachers, including a good number of pencil drawings in what may be called axonometric geometry.

SECOND SESSION OF THE ASSOCIATION.

(4) "Vectorial differential geometry" by Professor LOUIS BRAND, University of Cincinnati.

(5) "The fitting of Eulerian spirals to intersecting straight lines" by Professor E. R. SMITH, Iowa State College.

(6) "The place of mathematics in the Antioch program" by Dean P. C. NASH, Antioch College.

Discussion opened by Professor M. A. NORDGAARD, Antioch College.

(7) "Formation of nth order determinants with integral elements and with the value unity" by Professor HARRIS HANCOCK, University of Cincinnati.

(8) "A research interest in problems in the calculus of variations" by Professor MARY EMILY SINCLAIR, Oberlin College.

(9) "A solution of the cubic in the case of imaginary roots" by Professor OTTO DUNKEL, Washington University.

4. The first part of Professor Brand's paper dealt with various applications, to plane differential geometry, of the theorem on the derivative of the position vector of a moving point with respect to the arc it describes. Passing on to the geometry of twisted curves, kinematical concepts were introduced by regarding a curve as described by a point moving with unit speed. The curvature and torsion were then defined as the components of the angular velocity vector of the moving trihedral on the positive binormal and tangent. This definition led at once to the Frenet-Serret formulas, all three having precisely the same form. The curvature of surfaces was then treated by means of the derivative dyadic of the positive unit normal vector to the surface. This dyadic was determined directly by finding the derivative of the unit normal vector in three mutually orthogonal directions. The first and second scalar invariants of this dyadic give the mean and total curvature of the surface. By expressing the dyadic in an alternative form, two simple vector relations were obtained from which the mean and total curvature could be computed at once in terms of the fundamental quantities of the surface.

5. Euler discussed in one of his papers the equation of the spiral for which the curvature at any point is proportional to the length of the arc. The intrinsic equation of such a spiral may be written in the form $K = 2a^2s$ where K is the curvature at any point, s the length of the arc, and a the arbitrary constant. Professor Smith gave the cartesian equations for the curve in the usual form which may be found in "Topics for Club Programs," in the MONTHLY (*1918*, 276–282). He also pointed out that a spiral of this type is usable in certain types of highway curves; if the change of direction and the minimum radius of curvature permitted are given, it is possible to compute all the constants required in fitting the spiral to the highway. In this connection attention may be called to the equivalent presentation by George Paaswell in this MONTHLY (*1918*, 267–268).

6. According to Dean Nash there are three purposes sought in the Antioch plan: (1) to discover the genius in mathematics, the person who will go on to further study and will appreciate mathematics for its own sake as a field for deep study; (2) to call to the attention of the general student the advantages of mathematical concepts, to give him a habit of making generalizations and then dealing accurately and definitely with these, to give him the points of view brought out by the calculus; (3) to give to students a knowledge of mathematics as a tool for use in courses in physics, chemistry and engineering.

At Antioch every freshman is asked to take one year of mathematics, one half of the college course of six years is spent in actual contact with industry, with the practical hard facts of life as it really is; the other half of the course,

dovetailing in alternating five-week periods with the industrial part, is devoted to college studies. Only about one-half of this time is spent in specialization; for example, the accountant takes only about half of his academic courses in the field of business, and the other half in subjects outside the field of his direct specialized interest. Again the students in engineering are given only thirty-four courses in the fundamentals of mathematics, physics, chemistry, thermodynamics, etc.; the more technical courses are omitted in favor of twenty-six courses in literature, sociology, biology, psychology, philosophy, etc. The detailed technique of engineering can then be learned on the job or by special study later on.

In deciding upon the important fields of human knowledge with which every Antioch student is to have an acquaintance, mathematics is included partly for its own sake and partly for its value as a tool. The college recognizes the responsibility that comes in requiring mathematics of every student. He must first be convinced of the value of mathematics in outlining methods of generalization and in dealing beautifully and rigorously with these generalizations. Then the content of the course must be made intelligible and interesting to the student; it must be correlated with his experience as much as possible. While, of course, no text-book can do this alone, Griffin's "*Mathematical Analysis*" was mentioned as a text-book that aids greatly in this plan. Next the course in mathematics must be made worth while for all students whether they are to be journalists, artists, merchants, or engineers.

In such a heterogeneous group as is found here it is natural to meet with all kinds of mathematical preparation and ability, consequently on the opening day of college, before the freshmen register, they are given a mathematical placement examination, divided into three parts, arithmetic, algebra and geometry, twenty minutes being allowed for each. The students are then divided into six sections according to their marks in this examination. The lowest section uses a strong secondary text-book in general mathematics, and is not given credit for this first course. This one section proves to be the greatest problem of all six. Some of these students leave college during the year because they are similarly deficient in other subjects and are not good college material. Of those who remain, the weak ones keep on for the full year in this preparatory course, while the brighter ones change to the beginning college course at midyear. Whichever course is taken the students are not required to carry on more than the year's work, unless they are to specialize in engineering or science. If they are competent to do college work at all, this year's foundation work has given them some mathematical power, limited it is true, but nevertheless sufficient for their needs in subsequent courses. A minor amount of sectioning is again done at the beginning of the second semester according to progress made in the first semester, but the original placement examination separates them quite accurately for the year's work. Since in the lowest and the highest of the six sections there is a greater difference in abilities than in the middle sections, these two sections are necessarily smaller in number.

Many changes have been made in the organization of this course in the three years in which it has been tried; for instance, the scientific and engineering stu-

ents were formerly separated from those who expected to take only one year of mathematics. This policy has been changed and in the future there will be in the highest section a predominance of engineers, but a scattering of girls who expect to be journalists, or boys who want a general business training, and others. The course has been developed far enough so that the administration feels fairly confident that a year of required mathematics is of great benefit to nearly every student.

In his discussion Professor Nordgaard said that one of the articles of faith in the Antioch program is that the intellectual interests usually found in the case of educated persons should be represented in the college curriculum. On this ground all students are required among other subjects to take a course in mathematics. This also helps the student with innate liking and capacity for mathematics to find himself and to continue further; thus at Antioch one-third of the sophomore class takes mathematics as an elective. Antioch also aims to show to the student the place that each major study has held in the world's thought and in the up-building of our civilization. This is accomplished (1) by recounting the history of the different subjects, (2) by showing the uses made of mathematics, for example, in the arts and sciences, (3) by close coördination with the departments of chemistry and physics. The coöperative plan has distinct advantages. While the student does not relax as in an ordinary vacation, some of his mathematical concepts mature and clarify during the working period and the applications of mathematics are often found in the course of the student's work, so that the psychological effect is most salutary.

While there was not sufficient opportunity to go into the details of classification of students, Professor Nichols stated that Louisiana State University is making some attempt to group students according to their ability, following the ideas of Dr. Seashore. Last year the plan met with satisfactory results, but this year, due to the great increase in attendance at the University, the ratio of students to teachers has been entirely too large for satisfactory grading in a short time. A division of the students into two grades was made at the end of the first six weeks on the basis of written tests and the regular recitations, each teacher following his or her own selection of questions and methods of grading. Subsequent results have shown that the basis of grading was not sufficiently uniform, but that it can probably be improved by a closer coöperation of those who grade the students. No extra topics are being taken by the strong sections, more intensive work being done by them.

7. Let $e_1, e_2, \cdots, e_n$ be any n integers, and let d_2 be the G.C.D. of e_1 and e_2. Further let d_3 be the G.C.D. of e_1, e_2, e_3 and in general suppose that d_n is the G.C.D. of $e_1, e_2, \cdots, e_n$. It is known that two integers x_1, y_1 may be found such that $e_1x_1 - e_2y_1 = d_2$, or

$$\begin{vmatrix} e_1, & e_2 \\ y_1, & x_1 \end{vmatrix} = d_2.$$

A still more general theorem was proved by mathematical induction by Professor

Hancock. At the same time integers $t_1, t_2, \cdots, t_n$ were found such that $t_1e_1 + t_2e_2 + \cdots + t_ne_n = 1$. A different proof is given among others by Bianchi, *Lezioni sulla Teoria dei Numeri Algebrici*, pp. 1–7.

8. To illustrate unfinished problems in the calculus of variations, some of them dating from the days of Euler and the Bernoullis, two related problems were sketched by Professor Sinclair, both experimentally realized by Plateau with liquid films. The surface of revolution of minimum area has an absolute minimum, either an open catenoid within geometrically determinable limits of stability or simply two circular discs. The one-variable end-point case has been studied, and a compound discontinuous solution which is experimentally realizable. Questions still open relate to variable end points.

The second and related problem sketched in this paper is also that of a surface of revolution of minimum area but with prescribed volume. The generators of the surface are representable by elliptic integrals. Lagrange showed the surfaces to have constant mean curvature. Delaunay showed them to be roulettes of foci of conics. Howe's discussion of the Jacobi condition led to geometric results for limits of stability only in special cases. Moreover, the case of variable end points has been neglected, and a known compound discontinuous solution has not been treated mathematically.

There is noted by Boyes a connection of these problems with the formation of drops from a cylindrical column of liquid, and with the forming of the sticky beads of a spider's web. Cell division is a more or less related problem, as are other questions involving surface tension.

Miss Sinclair gave a history of the problems and indicated what parts of these are as yet unsolved. It is hoped that her presentation may appear in full in the MONTHLY, as it is the first outgrowth of the plan begun at the Toronto meetings, suggesting possible lines of research.

For the Committee on Carus Monographs Professor Slaught announced that the first monograph is to be an exposition of the calculus of variations by Professor G. A. Bliss of the University of Chicago.

9. The cubic equations considered by Professor Dunkel are in the form in which the second term containing the square of the unknown has been removed by a linear transformation. If the equation has three real roots, a comparison of the equation with the formula for $\sin 3\theta$ gives a known trigonometric solution which is very convenient for the simultaneous computation of the three roots. Professor Dunkel shows in this paper that an equally convenient and simple solution is given in the case of imaginary roots by a comparison with the similar formulas for $\sinh 3\theta$ and $\cosh 3\theta$ in the two cases which have to be separated in the treatment. This is such an obvious procedure that it is not likely that it is unknown,[1] but it appears, at least, to have escaped the attention of writers, as

[1] Professor R. C. Archibald points out that the value of such functions in this connection was recognized by J. H. Lambert in his *Zusätze zu den logarithmetrischen und trigonometrischen Tabellen zur Erleichterung und Abkurzung der bey Anwendung der Mathematik vorfallenden Berechnung*, Berlin, 1770, Chapter 1, art. 6. Among numerous references supplied by Professor Archibald are the following: J. F. W. Gronau, *Auflösung der kubischen Gleichungen durch trigonometrische*

it does not appear in the usual texts. Since tables of hyperbolic functions are very accessible, this solution should receive attention in preference to Cardan's solution or the complicated methods which are sometimes given for the determination of the imaginary roots. A simple transformation carries the solution by hyperbolic functions into the usual solutions by trigonometric functions (see *Lehrbuch der Algebra*, Weber, vol. I, pp. 391–393), but the trigonometric solution thus obtained is not nearly so simple as the one by the hyperbolic functions, and, in fact, it is not much better than the solution by Cardan's formula.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY, AND SECTION A OF THE AMERICAN ASSOCIATION.

(10) "American mathematics during three quarters of a century" by Professor G. A. MILLER, University of Illinois, retiring vice-president of Section A.

(11) "On the equation of the eighth degree" by Professor A. B. COBLE, University of Illinois, retiring chairman of the Chicago Section of the Society.

(12) "Algebras and their arithmetics" by Professor L. E. DICKSON, University of Chicago, by invitation of the Mathematical Association and the Chicago Section of the Society.

10. Professor Miller's address appeared in *Science* for January 4, 1924.

11. Professor Coble's address is to be printed in the *Bulletin* of the American Mathematical Society.

12. The object of Professor Dickson's address was to present the point of view and leading conclusions of a new general theory of the arithmetics of algebras. The initial theory presented by Lipschitz in 1886 was extremely complicated, while a successful theory was first obtained by A. Hurwitz in 1896, with further work by Du Pasquier in 1909, 1916, and 1920. These theories failed in various respects to cover all possible fields of the general subject. Through a new and happy definition, Professor Dickson has for the first time been successful in generalizing the subject so that it appears to bring to its final state of completeness the possible extensions of the oldest concept in mathematics, *i.e.*, the concept of integers. Owing to the outstanding character of this paper, a great deal of interest was aroused because of its possible relation to the Cincinnati Prize of $1,000.00 offered through the American Association for a "notable contribution to Science presented in the programs of the Cincinnati meeting." With complete unanimity the Trustees of the Association and the Committee of Section A of the American Association joined with the three secretaries in recommending this strongly to the Committee on Award. Subsequently the announcement was made that the award had been made to Professor Dickson.

Funktionen der Kreises und der Hyperbel, Danzig, 1861; W. Matzka in *Archiv der Mathematik und Physik*, vol. 37, 1861, pp. 399–419; J. A. Grunert, ibid., vol. 38, 1862; C. A. Laisant, *Essai sur les Fonctions Hyperboliques*, Paris, 1874, pp. 85–86; J. McMahon, *Hyperbolic Functions*, New York, 4th ed., 1906, pp. 45–46.

Meeting of the Board of Trustees of the Association.

Seven members of the Board were present. The following fifty-one persons and three institutions, on applications duly certified, were elected to membership:

To Individual Membership.

R. P. Agnew, A.B. (Allegheny). Grad. Asst., Iowa State Coll., Ames, Ia.
Wealthy Babcock, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.
O. L. Barnett, B.S. Instr., High School, Provo, Utah.
V. I. Benander, A.B. (Harvard). Instr., Case School of Appl. Sc., Cleveland, Ohio.
L. M. Berkeley, A.M. (Virginia). 36 W. 91st St., New York, N. Y.
S. F. Bibb, M.S. (Chicago). Instr., Univ. of N. Dak., Grand Forks, N. Dak.
W. M. Bond, A.M. (Columbia). Instr., Coll. of City of N. Y., New York, N. Y.
C. F. Bowles, A.M. (Harvard). Instr., S. Dak. School of Mines, Rapid City, S. Dak.
G. R. Bushyager, B.S. (Allegheny). Instr., Penn. State Coll., State College, Pa.
C. J. Cardin, M.E. (Lehigh). Instr., Washington & Jefferson Coll., Washington, Pa.
I. S. Carroll, A.M. (Columbia). Instr., Syracuse Univ., Syracuse, N. Y.
J. E. Case, A.M. (Gonzaga). Prof., St. Mary's Coll., St. Mary's, Kans.
Annie C. Clark (Mrs. Theron Clark), Ph.B. (Brown). Instr., Bucknell Univ., Lewisburg, Pa.
S. R. Cruse, M.S. (Arizona). Asst. Prof., Univ. of Arizona, Tucson, Ariz.
W. F. Dantzscher, Senior, Columbia College, Livingston Hall, Columbia Univ., New York, N. Y.
J. S. Elston, A.B. (Cornell). Asst. Actuary, Life Dept., Travelers Ins. Co., Hartford, Conn.
Ruth M. Feagin, B.S. (Alabama). Head of Dept. of Math., Martin Coll., Pulaski, Tenn.
P. J. Federico, B.S. (Case School). Asst. Examiner, U. S. Patent Office, Washington, D. C.
H. K. Fulmer, A.M. (Columbia). Instr., Ga. School of Tech., Atlanta, Ga.
F. J. Gerst, A.M., M.S. (St. Louis Univ.). Asst. Prof., St. Xavier Coll., Cincinnati, Ohio.
V. D. Gokhale, Ph.D. (Chicago). Asso. Prof., Univ. of the Philippines, Manila, P. I.
E. H. Hadlock. Stud. Asst., Univ. of Maine, Orono, Me.
W. L. Harris, B.S. (Georgetown Coll.). Grad. Asst., Ia. State Coll., Ames, Ia.
C. R. Hillard, A.B. (Franklin & Marshall). Head of Dept. of Math., Coll. of the Ozarks, Clarksville, Ark.
H. B. Huntley, A.B. (Harvard). Instr., N. H. Coll., Dover, N. H.
W. H. Kurzin, M.S. (Chicago). Instr., Crane Jr. Coll., Chicago, Ill.
G. Farley Lee, A.M. (Columbia). Prof., Judson Coll., Marion, Ala.
P. S. Marthakis, M.S. (Utah). Teacher, West High School, Salt Lake City, Utah.
F. J. McMackin, Ph.D. (Columbia). Instr., Cooper Union, New York, N. Y.
Sister Mary McNelly, A.M. (Catholic Univ.). Teacher, Our Lady of the Lake Coll., San Antonio, Tex.
G. E. Moore, M.S. (Illinois). Prof., Hanover Coll., Hanover, Ind.
M. E. Morrissy, A.M. (Dubuque). Instr., Marquette Univ., Milwaukee, Wis.
W. J. Newlin, A.M. (Amherst; Harvard). Prof. Math. and Phil. and Secy. of the Faculty, Amherst Coll., Amherst, Mass.
Helen E. Nugent, B.S. (Ia. State Coll.). Teacher, High School, Algona, Ia.
Leander Olsen, B.S. (Utah Agric. Coll.). Instr., High School, Moroni, Utah.
George Rosengarten, C.E., Ph.D. (Penna.). Prof. of Physics, W. Phila. Boys High School, Philadelphia, Pa.
W. E. Roth, A.B. (Wisconsin). Teacher, High School, West Allis, Wis.
J. B. Scarborough, Ph.D. (Johns Hopkins). Asst. Prof., U. S. N. A., Annapolis, Md.
Mary E. Schofield, A.M. (California). Head of Dept. of Math. and Sc., Texas Presbyter. Coll., Milford, Tex.
R. C. Shellenberger, A.M. (Michigan). Head of Dept. of Math., City Schools and Jr. College, Bay City, Mich.
Jessie M. Short, A.M. (Carleton). Instr., Reed Coll., Portland, Ore.
H. W. Smith, B.S.C.E. (Georgia). Head of Dept. of Math., High School, Gainesville, Ga.
F. W. Sparks, A.M. (Southwestern), M.S. (Chicago). Asst. Prof., A. & M. Coll. of Texas, College Station, Tex.
Antoinette Spies, A.M. (Fordham). Instr., Coll. of the Sacred Heart, Manhattanville, New York, N. Y.

EUGENE TAYLOR, A.M. (DePauw). Prof., Univ. of Idaho, Moscow, Idaho.
JOHN TYLER, A.M. (William & Mary). Asst. Prof., U. S. N. A., Annapolis, Md.
E. M. VAN ORDEN, B.S. (Utah). Instr., High School, Richmond, Utah.
E. F. WEINBERG, C.E. (Manhattan). Prof., Rollins Coll., Winter Park, Fla.
J. F. WELCH, B.S. (Ohio State). Asso. Prof., La. State Univ., Baton Rouge, La.
G. L. WINKELMANN, A.M. (St. John's). Head of Dept. of Math., St. John's Univ., Collegeville, Minn.
META A. WOOD, A.M. (Wisconsin). Instr., Synodical Coll., Fulton, Mo.

To Institutional Membership.

THE COLLEGE OF THE OZARKS, Clarksville, Ark. Prof. C. R. Hillard, Official Repr.
THE NEW JERSEY COLLEGE FOR WOMEN, New Brunswick, N. J. Dean Mabel S. Douglass, Official Repr.
COLLEGE OF MOUNT ST. VINCENT, Mount St. Vincent-on-Hudson, N. Y. Sister Josephine Rosaire, Official Repr.

The following have been appointed associate editors of the MONTHLY for the year 1924:

N. H. ANNING	B. F. FINKEL	C. N. MILLS
R. W. BURGESS	TOMLINSON FORT	F. D. MURNAGHAN
OTTO DUNKEL	D. C. GILLESPIE	D. E. SMITH
H. J. ETTLINGER	C. F. GUMMER	V. H. WELLS

Professor A. J. KEMPNER was appointed as the third member of the Committee on Official Journal, the vacancy being caused by the election of Professor Coolidge as a vice-president. The Secretary and Dean T. M. FOCKE of Case School of Applied Science were appointed the representatives of the Association on the Council of the American Association for 1923–1924 meetings. The resignation of Professor Karpinski as Librarian was accepted and Professor R. B.MC CLENON of Grinnell College was appointed to this position. It was voted that the appointments of Professor Cairns as Secretary-Treasurer, Professor Slaught as Manager, Professor Ford as Editor-in-chief, Professor Kempner as third member of the Committee on Official Journal, Professor McClenon as Librarian, Professor Sinclair as Assistant Librarian, and Professor Yeaton as Assistant Secretary should be regarded as permanent rather than annual appointments. Professors R. C. ARCHIBALD, E. H. MOORE and J. W. YOUNG were appointed as delegates of the Association on the American Section of the International Mathematical Union for the Toronto meeting next August. Dr. A. B. CHACE was appointed to the vacancy in the Board of Trustees for the term ending January 1926, caused by the election of Professor Jackson as a vice-president. It was voted to hold the annual meeting in December 1924, in Washington, D. C., in affiliation with the American Association. It was voted to set aside $1000.00 as the nucleus of a permanant endowment fund. This amount has been put into the form of U. S. Treasury Savings Certificates at $4\frac{1}{2}\%$ compounded semi-annually.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership by the Board. He reported also the death of the following members:

T. M. BLAKSLEE, Retired, Des Moines College (January 30, 1923);
S. D. KILLAM, Professor of mathematics, University of Alberta (July 22, 1923);
MALCOLM MCNEILL, Professor of mathematics, Lake Forest College (October 5, 1923);
A. D. PITCHER, Professor of mathematics, Western Reserve University (October 5, 1923);
C. B. WALSH, Headmaster, Woodmere Academy (July 13, 1923);
A. G. WEBSTER, Professor of physics, Clark University (May 15, 1923);
C. B. WILLIAMS, Dean and professor of mathematics, Kalamazoo College (September 1, 1923);
D. T. WILSON, Professor of astronomy, Case School of Applied Science (October 12, 1923).

The election of officers for the year 1924 was conducted by mail and in person at this meeting, as provided in the By-Laws, the tellers, Professors R. B. Allen and Bird M. Turner, reporting the result of the balloting as follows:

For President: L. P. Eisenhart, 248 votes; H. L. Rietz, 251 votes.

For Vice-Presidents: Clara L. Bacon, 227 votes; A. B. Chace, 225 votes; J. L. Coolidge, 249 votes; Dunham Jackson, 275 votes.

For additional members of the Board of Trustees (to serve until January 1927): R. D. Carmichael, 363 votes; E. R. Hedrick, 350 votes; G. H. Light, 145 votes; C. N. Moore, 230 votes; R. E. Moritz, 194 votes; W. W. Rankin, Jr., 100 votes; D. E. Smith, 300 votes; E. J. Wilczynski, 298 votes.

The following were accordingly declared elected:

President: H. L. RIETZ, University of Iowa.

Vice-Presidents: J. L. COOLIDGE, Harvard University; DUNHAM JACKSON, University of Minnesota.

Additional members of the Board of Trustees: R. D. CARMICHAEL, University of Illinois; E. R. HEDRICK, University of Missouri; D. E. SMITH, Columbia University; E. J. WILCZYNSKI, University of Chicago.

The following resolution was offered by Professor Fort: "Resolved, That a committee be appointed to take under advisement the general problem as to how the Association can be of service to departments of mathematics in the various colleges, and in particular to formulate standards to which departments of mathematics should conform." This resolution was adopted unanimously, and the following committee was later appointed by President Rietz: Professor CARMICHAEL, chairman; Professors R. C. ARCHIBALD, G. C. EVANS, MARY EMILY SINCLAIR, E. R. SMITH, and J. W. YOUNG.

A letter was received for the Association and the American Mathematical Society from Professor Shizumo Sano, chairman of the Physico-Mathematical Society of Japan, expressing their gratitude for the sympathy of the mathematicians of America in the calamity caused by the earthquake of September first, as voiced in a letter sent from the Vassar meetings, and saying that while none of the members of that Society lost their lives during the earthquake there was a large loss in the equipment for work pertaining directly to mathematics, physics, and allied sciences, from which they are gradually recovering.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 4, 1923.

RECEIPTS.

Balance Dec. 5, 1922		$ 6,799.73
1922 indiv. dues	$ 285.60	
1922 instit. dues	21.00	
1922 subscriptions	7.00	
Sale of Register	2.75	
1923 indiv. dues	5,450.78	
1923 instit. dues	611.00	
1923 subscriptions	785.80	
Contributions to 1923 expenses	409.00	
Initiation fees	533.00	
Life membership fees	105.34	
Sale copies MONTHLY	83.61	
Sale of reprints	3.00	
Advertising	696.00	
Received for authors' corrections	10.00	
Interest Oberlin Savgs. Bk.	131.69	
Interest Peoples Bkg. Co.	104.96	
Interest U. S. Bonds	53.74	
Total 1923 receipts		$ 9,294.27
Total assets to the end of 1923 business		$16,094.00
Total expenditures		11,027.59
Balance to the end of 1923 business		$ 5,066.41
Received on 1924 business		461.15
Book balance Dec. 4, 1923		$ 5,527.56

EXPENDITURES.

Publisher's bills (Jan.–Mar., Aug.–Dec., 1922; Jan.–Oct., 1923)		$ 7,258.09
Register		593.57
Manager's office		23.64
Editor-in-chief's office, 1922 vol.		288.33
Editor-in-chief's office, 1923 vol.		460.53
Other editors' postage		7.40
Committee on Membership		498.06
Secretary-Treasurer's office:		
Postage	$265.10	
Bond	5.00	
Safety deposit	4.00	
Office supplies	35.72	
Express, tel., etc.	49.34	
Clerical work	629.25	
Printing	374.50	
Cambridge meeting	80.00	
Vassar meeting	77.67	
Paid copies of MONTHLY	80.90	
Refund on subscriptions	34.55	
Library expense	57.25	
Paid to sections from initiation fees	54.69	
		$ 1,747.97
Annals subvention		150.00
Total expenditures		$11,027.59
Cash on hand		$ 59.81
Checking account		535.53
Oberlin Savgs. Bk. account		2,051.35
Peoples Bkg. Co. account		1,880.87
Liberty Bond		500.00
U. S. Treasury Note		500.00
Bank balance Dec. 4, 1923		$ 5,527.56

Of the funds on hand $138.98 is held in a separate Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1924.

Aside from the above-mentioned funds on hand the first payment of $1,200.00 on the Carus Fund amounts with the interest accruing at 4 per cent., compounded quarterly, to $1,286.54.

When the accounts were closed December 4, 1923, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1923 the following items:

Bills Receivable.	
1923 indiv. dues unpaid	$120.00
1923 instit. dues unpaid	21.00
Advertising	100.00
Interest U. S. Treasury Note	11.00
	$252.00

Bills Payable.	
(Either paid in December or estimated)	
Publisher's bills (Nov., Dec., 1923)	$1,525.00
Manager's office	40.00
Editor-in-chief's office	150.00
Other editors' postage	30.00
Committee on Membership	100.00
Secretary-Treasurer's office	200.00
Annals subvention	250.00
Initiation fees due to sections	300.00
Printing annual ballots, programs, etc.	175.00
Life Membership Fund	139.00
	$2,909.00

If to the balance on 1923 business shown in this report, $5.066.41, there be added the bills receivable, $252.00, and there be subtracted the estimated amount of bills payable, $2,909.00, there results an estimated final balance on 1923 business of approximately $2,400.00. The corresponding estimated final balance one year ago on 1922 business was $1,800.00. Thus, aside from the $400.00 which Dr. Chace has generously given for the second year, there is approximately a gain of $200.00 on the year's business. This narrow margin of profit has been gained through the undesirable expedient of combining two months' issues of the MONTHLY into a single number at various times in 1922 and 1923; this was adopted in order to bring up to date the tardy printing schedule and it brought with it some financial advantage which, while desirable in itself, cannot be repeated. The experience of other scientific journals indicates that printing rates at present are quite unstable and your officers are concerned that the Association's financial basis may be more secure than is as yet the case and that with increasing income further important lines of activity may be undertaken.

W. D. CAIRNS, *Secretary-Treasurer.*

RESEARCH FELLOWSHIPS IN MATHEMATICS.

The National Research Council in 1919 was entrusted by the Rockefeller Foundation with the expenditure of an appropriation of half a million dollars within a period of six years for promoting fundamental research in physics and chemistry, primarily in educational institutions in the United States. Recently this grant has been renewed for a second period of five years, and has been increased to $625,000, to take effect July 1, 1925. At the same time these fellowships were extended to include mathematics, this provision to take effect at once, so that applications for mathematical fellowships will be considered at the April meeting of the Fellowship Board. This Board, hereafter, is to contain two mathematicians.

The primary feature of the plan is the initiation and maintenance of a system of National Research Fellowships which are to be awarded by the National

Research Council to persons who have demonstrated a high order of ability in research, thus enabling them to conduct investigations at educational institutions which make adequate provision for effective prosecution of research in physics, chemistry and mathematics. The results which are expected to follow are: (*a*) the benefit of a scientific career opened to a larger number of able investigators, (*b*) the opportunity for more thorough training of young men and women in research, (*c*) resultant extension of knowledge of the fundamental principles of physics, chemistry and mathematics, and (*d*) the creation of more favorable conditions for research in the educational institutions of this country.

These Fellowships are administered by a Board representative of the different sciences involved and are awarded, after careful consideration, to such as may be found entitled to them. The fundamental requirements of an applicant include a Ph.D. degree or its equivalent, evidence from the past of high ability and good character, promise for the future backed by the opinions of men closely associated with him in scientific work, and finally the general nature of the problem submitted for investigation.

Applicants are permitted to select, with the approval of the Board, the institutions at which they may choose to work. Fellowships are awarded for a period of twelve months with a minimum stipend of $1,800 for men without dependents, and $2,300 for men with dependents. Fellows may be re-appointed for a second or a third year with an advance in stipend as determined by the Board in each particular case. Fellows are expected to devote their entire time to research except that during the college year they may, at their option, give a portion of their time—in general not more than one-fifth—to teaching deemed to be of educational value to themselves, or to attendance on advanced courses of study. The results of their investigation must be made available to the public without restriction.

THE PRESENT STATE OF THE DIFFERENCE CALCULUS AND THE PROSPECT FOR THE FUTURE.[1]

By R. D. CARMICHAEL, University of Illinois.

1. **Remarks concerning the present state of the difference calculus.** It is probably correct to say that the modern development of the difference calculus began with a memoir by Poincaré published in 1885 in the seventh volume of *The American Journal of Mathematics*. He considered a difference equation of the form

$$f(n+k) + p_{k-1}(n)f(n+k-1) + \cdots + p_1(n)f(n+1) + p_0(n)f(n) = 0,$$

where the independent variable n runs over the set of positive integers and where the coefficients $p_i(n)$ are functions of n such that

$$\lim_{n=\infty} p_i(n) = A_i, \qquad i = 1, 2, \cdots, n; \; A_0 \neq 0.$$

[1] Retiring presidential address before The Mathematical Association of America. Cincinnati, Dec. 27, 1923.

In case the characteristic algebraic equation

$$\rho^k + A_{k-1}\rho^{k-1} + \cdots + A_1\rho + A_0 = 0$$

has k roots ρ_i no two of which are equal in absolute value, then, if $f(n)$ denotes any particular solution of the given difference equation, the quotient $f(n+1)/f(n)$ approaches a limit as n approaches infinity and this limit is one of the roots ρ_i of the characteristic algebraic equation. This is an important theorem established by Poincaré. Investigations more or less closely related to this path-breaking memoir by Poincaré have been carried out by Pincherle, Ford, Nörlund, Perron, Horn, Erb, Van Vleck, and others.

These researches have yielded important applications in the study of convergent series and continued fractions and in the integration of linear differential equations. In these researches, so far as difference equations are involved, one is concerned principally with the asymptotic character of the solutions for large *integral* values of the independent variable. A much more important and characteristic problem of the difference calculus first came into view in the study of the analytic properties of the solutions for general real or complex values of the independent variable; this takes its most interesting form in the case when functions of a complex variable are treated. This type of research was initiated almost simultaneously by Carmichael, Birkhoff, Galbrun, and Nörlund about the end of the first decade of our century. The investigations carried out in this connection have brought into notice a large and important class of functions having properties in many respects analogous to those of the classic gamma function, the latter indeed affording an example of this class of functions. It is a historical fact of some interest that a function of such importance as the gamma function was frequently employed in analysis for several generations before it came to be recognized as but the simplest instance of a large class of useful functions lying rather close to hand but as yet undiscovered.

The genesis of scientific investigation is often a matter of considerable interest. Perhaps I may be pardoned for saying a word here about the origin of the American contributions to the theory of the difference calculus, especially since my dissertation holds a place in the early part of this history. The first impulse in America, and the first in the world so far as I know, toward the development of the difference calculus from the point of view of general function theory, was given at the University of Wisconsin in 1909 in the lectures of E. B. Van Vleck. This fact has been put on record by G. D. Birkhoff[1] in the following words: "To the best of my knowledge, the importance of the functional standpoint in the field of difference equations was emphasized first by Van Vleck in an inspiring series of lectures given at the University of Wisconsin in the spring of 1909, in which he conjectured the existence of sets of solutions analytic on either the left or the right side of the complex plane."

In 1907 Birkhoff received his degree at the University of Chicago. In the two years 1907–1909 he was an instructor at the University of Wisconsin; he

[1] *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 243.

took advantage of the opportunity afforded him to attend the lectures given there by Van Vleck and in this way became thoroughly conversant with the point of view upon which Van Vleck was insisting relative to the most promising direction in which to undertake the development of the difference calculus. In the fall of 1909 he went to Princeton University as a preceptor. At the same time I became a student at Princeton University, attending the lectures of Birkhoff on differential equations. In discussing with him possible subjects for my dissertation I learned of the nature of some outstanding problems in the difference calculus and especially of those whose importance had been insisted upon by Van Vleck in his Wisconsin lectures and I determined to undertake the establishment of general existence theorems for linear homogeneous difference equations for the case of functions of a complex variable.

That the power series did not afford a suitable tool for this investigation was evident from certain general considerations; it was therefore natural to try some form of the method of successive approximations. The formal aspects of this method were fairly obvious; but a certain obstacle in its application was early met with owing to the very difficult character of the problem of finite integration, a problem which had not been solved in a form suitable for use in the investigation in hand. The chief novelty of method in my investigation was in the development of a means of finding the finite integrals of certain classes of functions which had to be dealt with in the course of the argument. To this general problem of finite integration I shall return later on in the lecture.

A similar difficulty with finite integration had been met with a few years earlier by J. Horn and the investigation which he carried out was restricted in range by the lack of a suitable tool for finite integration, as he himself later stated;[1] the method of finite integration employed in my dissertation would have sufficed to overcome the main difficulty encountered by Horn. In the investigations of Birkhoff, published soon after my own, the theory of the solutions of linear homogeneous difference equations reached a state of development distinctly in advance of what had previously been attained. The essential improvement in method consisted in a certain fairly obvious extension of the means of finite integration introduced by me, the extension having the effect of giving the method a considerable increase in flexibility, an advantage entirely unforeseen by me at the time of writing my dissertation. Birkhoff also employed the method of successive approximations, giving it a different (and very convenient) form by the use of the matrix notation.

Two other basic investigations arriving at general existence theorems for linear homogeneous difference equations were carried out almost simultaneously with those due to Birkhoff and myself. Galbrun treated the problem by the aid of the Laplace transformation and the consequent reduction of the investigation to a dependence on known results concerning differential equations. Nörlund treated the problem by aid of factorial series; in some respects the method employed by him is more satisfactory than any other yet developed. I have never

[1] Crelle's *Journal*, 141 (1912), p. 183.

been certain that his method is the best possible. It has seemed to me to be probable that a generalization of the method of Nörlund, effected by means of the series treated in section 5 of this address, may afford a still more satisfactory method than that of Nörlund, especially for the important case in which the coefficients in the difference equation are rational functions.

A considerable development of the theory of linear difference equations has followed the basic investigations by the four mathematicians whose work we have just outlined, this having been carried out both by them and by others. Up to the present time but little progress has been made in the development of the theory of non-linear equations. Furthermore, even in the case of linear equations, we have so far had developed in detail the theory of only a few of the particular equations whose solutions may be of especial interest. The outstanding example is that of the equation $g(x+1) = xg(x)$ and the gamma functions which satisfy it. The theory of these functions was already well known before the development of general existence theorems for linear difference equations. There is much still to be done in the way of basic investigations of particular equations and of the development of a general theory of non-linear equations. Hardly a start has been made toward the development of a theory of partial difference equations.

An article by Nörlund, finished in April, 1922, and entitled "Neuere Untersuchungen über Differenzengleichungen," has been published in the *Encyklopädie der Mathematischen Wissenschaften*. This gives an adequate account of the present state of the difference calculus with precise references to the literature of the subject. A paper on the present state of the theory of difference equations had already been published by Nörlund in *Bulletin des Sciences Mathématiques* in 1921. The existence of these papers justifies me in giving but few explicit references in this address.

2. The rôle of the arbitrary elements. We have already indicated incidentally two of the reasons why the modern development of the difference calculus was so long delayed. It could not proceed far without the development of a fairly adequate theory of finite integration; and this is a problem of no little difficulty. The tool which was so successful in the development of the theory of differential equations, namely, the power series, is altogether inadequate for the difficulties encountered in the case of difference equations; power series are lacking in a certain flexibility requisite in any tool serving this purpose. But it seems certain that these difficulties—great as they are— would have been overcome if they had been resolutely faced. A certain important fact about difference equations probably operated to prevent for a long time a serious attack upon the problem afforded by them. This lies in the nature of the arbitrary elements.

To make the matter of arbitrary elements explicit let us consider the equation

$$f(x+k) + c_{k-1}(x)f(x+k-1) + \cdots + c_1(x)f(x+1) + c_0(x)f(x) = 0,$$

where the coefficients $c_i(x)$ are analytic at infinity. In terms of a suitably independent set $f_1(x), f_2(x), \cdots, f_k(x)$ of solutions of the given equation the general

solution is expressible in the form

$$f(x) = p_1(x)f_1(x) + p_2(x)f_2(x) + \cdots + p_k(x)f_k(x),$$

where $p_1(x)$, $p_2(x)$, $\cdots$, $p_k(x)$ are arbitrary periodic functions of period unity. In the case of the q-difference equation

$$f(q^k x) + c_{k-1}(x)f(q^{k-1}x) + \cdots + c_1(x)f(qx) + c_0(x)f(x) = 0, \qquad |q| \neq 1,$$

the general solution takes the form

$$f(x) = P_1(x)f_1(x) + P_2(x)f_2(x) + \cdots + P_k(x)f_k(x),$$

where $P_1(x)$, $P_2(x)$, $\cdots$, $P_k(x)$ are arbitrary q-periodic functions, that is, functions which satisfy the relation $P(qx) = P(x)$ and are otherwise arbitrary.

The difficulty which these arbitrary elements bring to light is of the following sort. In each case there is a significant region of the complex plane throughout which the solution is quite arbitrary, owing to the nature and place of the arbitrary elements in the solution. In a preliminary view of the situation one might well be led to conclude that little of interest is likely to arise from the theory of an equation whose general solution has so much of the arbitrary in it. It might not be easy to realize in advance that there is a ready means at hand for effectively reducing the rôle of the arbitrary elements; this is the more likely to be true in the case of the ordinary difference equation since the solution in this case is arbitrary in an infinite strip. It has seemed to me that the presence of these arbitrary elements in the general solution of difference equations probably operated to prevent the early development of an interest in them from the point of view of general function theory.

This could hardly have been true if the q-difference equation had been contemplated from the beginning as the ordinary difference equation had; for it is a relatively easy matter to see how to reduce the arbitrariness in the solution by confining attention to a certain class only of the solutions of the equation. Let us consider a q-periodic function $P(x)$, $|q| \neq 1$. Then $P(qx) = P(x)$. Let us suppose now that we restrict this function to be analytic at the point $x = 0$. It is easy to see that $P(x)$ must then be a constant. In view of this fact one may readily define a large class of linear homogeneous q-difference equations having a fundamental system of solutions each function of which is analytic at the point $x = 0$, while the most general solution *which is analytic at the point* $x = 0$ is expressible linearly in terms of the fundamental system, the multipliers being *constants*. The uncomfortable excess of arbitrariness is thus removed by means of a *descriptive condition* by which the admissible solutions are to be restricted.

In the case of the difference equation the matter is not so simple. Here the arbitrary elements are to be restricted by means of the asymptotic character of admissible solutions for certain defined methods of approach of the independent variable to infinity. This can not be made explicitly clear in short space; consequently the matter will be passed over with the general statement that it is possible to obtain the desired restriction of arbitrary elements by means of sets of limiting conditions at infinity.

Some important applications of the difference calculus, I believe, are yet to be made in connection with the theory of elliptic functions. As is well known, these functions are subject to a great variety of conditions arising from the addition theorems and results of a similar character. These may be particularized in many ways so as to give rise to difference equations, or q-difference equations, or to mixed equations of many different forms. The arbitrary elements in the general solution of these functional equations are often of such sort as to render apparently uninteresting the study of the general solution. But if one restricts the class of solutions, say by requiring analyticity at a certain point or in some other suitable way, one will often find that the solution is then restricted to be essentially the elliptic function from which the equation arose. We have had no systematic examination of the theory of elliptic functions from this point of view. I believe that such an investigation would serve a rather useful purpose in simplifying the exposition of certain topics in the theory of elliptic functions; this would be especially true in connection with the solutions of various types of functional equations, such, for instance, as the three-term functional equation for the Weierstrass σ-function.

3. The problem of finite integration. Perhaps the most important and the most difficult problem in the difference calculus is that of the determination of the solution of the equation

$$F(x+1) - F(x) = \varphi(x).$$

This is the problem of the finite integration of the function $\varphi(x)$. The existence of a solution is immediately obvious. The general solution is obtained by adding to a particular solution an arbitrary periodic function of period unity. The main problem of finite integration is to select from the infinite totality of particular solutions that one or those solutions which are characterized by the possession of suitably chosen fundamental properties. The difficulty of the problem consists in seeing how to select those properties and how to construct the solution or solutions possessing them. This fundamental problem has been extensively investigated by Nörlund. Besides the foregoing equation he studies also the following related equation:

$$G(x+1) + G(x) = \varphi(x). \tag{2}$$

He shows in particular the existence of principal solutions $F(x)$ and $G(x)$ of these respective equations defined by use of a method of summation analogous to that by means of which one obtains the solution of the equation

$$\frac{dH(x)}{dx} = \varphi(x)$$

in the form of a definite integral; and he denotes these solutions by the symbols

$$F(x) = S\varphi(x)\Delta x \qquad \text{and} \qquad G(x) = S\varphi(x)\Delta x$$

respectively. For these investigations we must refer the reader to Nörlund's

Encyclopædia article already mentioned and to the papers there cited; one should also examine the very recent paper by Nörlund in *The Transactions of the American Mathematical Society*, vol. 25.

A less important aspect of the problem of finite integration may be referred to here as setting certain additional problems which it seems desirable to have investigated. All these are connected with the so-called formal solutions of equations (1) and (2). To a considerable extent, but not completely, these have already been treated in the case of equation (1). It seems that much less has been done directly with equation (2), although the latter is in some respects rather simpler than equation (1).

Two formal solutions of equation (1) are given by the well-known direct sums to the right and to the left, respectively, namely, the solutions:

$$F(x) = -\varphi(x) - \varphi(x+1) - \varphi(x+2) - \cdots,$$
$$F(x) = \varphi(x-1) + \varphi(x-2) + \varphi(x-3) + \cdots.$$

The corresponding formal solutions of (2) are:

$$G(x) = \varphi(x) - \varphi(x+1) + \varphi(x+2) - \varphi(x+3) + \cdots,$$
$$G(x) = \varphi(x-1) - \varphi(x-2) + \varphi(x-3) - \varphi(x-4) + \cdots.$$

The formal series for $F(x)$ converge, and hence define an actual solution of (1), in case $\varphi(x)$ is analytic at infinity and vanishes there to at least the second order. The series for $G(x)$ converge if $\varphi(x)$ is analytic at infinity and vanishes there to at least the first order. Various modifications of the series for $F(x)$ have been employed, the modifications having been made with the purpose of extending the range of validity of the resulting series. Corresponding modifications of the series for $G(x)$ may also be made and with a like extension of the range of the validity of the resulting series. It appears to me that results of usefulness are likely to emerge from a systematic examination of the possible modifications of the foregoing formal solutions, those modifications being sought which will have the effect of giving a widened range of functions $\varphi(x)$ for which the resulting series shall actually afford a finite integral. In this connection one will probably need to examine summable divergent series as well as convergent series.

The well-known Euler-Maclaurin summation formula is obtained by writing (1) in the symbolic form

$$(e^D - 1)F(x) = \varphi(x)$$

by means of the Taylor expansion of $F(x+h)$ in powers of h (taken for $h = 1$), whence

$$F(x) = (e^D - 1)^{-1}\varphi(x),$$

where D denotes differentiation with respect to x, the explicit formula for $F(x)$ being gotten by expanding $(e^D - 1)^{-1}$ in powers of D and then operating with it term by term upon $\varphi(x)$. A similar formal solution of (2) may be obtained from the formula

$$G(x) = (e^D + 1)^{-1}\varphi(x).$$

Wedderburn[1] has obtained another formal solution of (1) by first expanding $(e^D - 1)^{-1}$ in partial fractions and then operating upon $\varphi(x)$ with the result. In a similar way a formal solution of (2) could be obtained by expanding $(e^D+1)^{-1}$ in partial fractions and then operating upon $\varphi(x)$ term by term with the result. I do not remember to have seen these formal solutions of (2) treated anywhere in the literature, but their formation is obvious in view of the known solutions of (1). It appears to me to be a matter of some interest to have a systematic analysis made of the possible modifications of these formal solutions and an investigation of the range of validity of the resulting series. It appears that the literature contains little (if anything) looking in the direction of these modifications or extensions of such formal solutions.

In case $\varphi(x)$ is an entire function and we write

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

we have a formal solution of (1) in the form

$$F(x) = \frac{a_0}{1} B_1(x) + \frac{a_1}{2} B_2(x) + \cdots + \frac{a_n}{n+1} B_{n+1}(x) + \cdots, \tag{3}$$

where $B_1(x)$, $B_2(x)$, $\cdots$ are the Bernoulli polynomials. This series is in general divergent. Appell (Liouville's *Journal*, (4) 7 (1891): 157–176) subtracts from $B_n(x)$ the first n terms of its trigonometric series, denoting the result by $\Psi_n(x)$. Then he shows that the series

$$F(x) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} \Psi_n(x) \tag{4}$$

converges uniformly in a given bounded region and defines a solution $F(x)$ of (1) which is an entire function. A similar method can be applied to the determination of a solution of equation (2).

In case $\varphi(x)$ is formally or actually represented by an expansion in factorial series or by an expansion in binomial coefficients, a term by term finite integration will yield a formal or an actual solution of the problem of the finite integration of $\varphi(x)$. For these expansions a similar problem may be proposed.

The way in which the divergent series in (3) is transformed into the convergent series in (4) may be suggestive of suitable methods of dealing with the other problems characterized in this section.

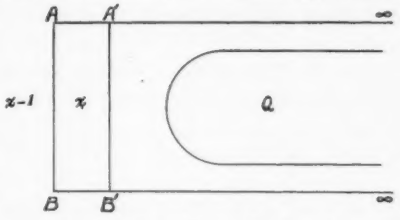
Two other related formal solutions of ( ) will be mentioned briefly; they are the ones first employed in my dissertation and afterwards extended in Birkhoff's memoir. [Corresponding formal solutions of (2) exist, but they have probably not yet made their appearance in the literature.] They are expressed by means of contour integrals. Let $\varphi(x)$ be analytic outside of a region Q extending to the right and lying within a finite distance of the positive half of the axis of reals, as in the figure: then if the contour L, $L = \infty\, AB\, \infty$ is drawn so as to enclose Q,

[1] *Annals of Mathematics*, vol. 16 (1914): 82–85.

and passes between $x - 1$ and x in the manner shown in the figure, the function

$$F(x) = \int_L \frac{p(x, z)\varphi(z)dz}{e^{2\pi\sqrt{-1}(x-z)} - 1}$$

formally satisfies (1) as may be seen by means of the theory of residues, provided that the function $p(x, z)$ is an analytic function of x and of z for all finite values of these variables while $p(x + 1, z) = p(x, z)$ and $p(x, x) = 1$. In a similar manner a contour integral to the left may be used to define another formal solution of (1), the corresponding function $p(x, z)$ being such that $p(x, x) = -1$; it is the latter which is handled explicitly in my dissertation, but only for the case $p(x, z) \equiv -1$. The more general form involves the extension made by Birkhoff. No general investigation has been made of the range of validity of these formal solutions. They are shown to be effective in the investigations in which they have been employed. It appears to me to be distinctly desirable to have a systematic analysis made of the range of validity of these formal solutions as actual solutions of the problem of finite integration. In this connection one would investigate also the various possible modifications and the related solutions, such (for instance) as those earlier employed by Guichard and Weber.[1]



4. Numbers defined by recurrence relations. An application of the difference calculus in the theory of numbers, the importance of which has not been generally recognized, is attached to equations in which the independent variable runs over the set of positive integers alone. Many of the classic theorems in the theory of the divisibility and primality of integers appear as special cases of theorems relating to sequences of integers defined by means of recurrence relations; and not a few rather novel results also emerge in this connection. The importance of this method of investigation was emphasized by Lucas in a series of investigations the principal results of which were brought together in his memoir in the first volume of *The American Journal of Mathematics*. The more general features of this method I have set forth in some fulness in a recent memoir.[2] For some years it has appeared to me that there is good reason to think that Fermat made frequent use of sequences of integers defined by recurrence relations and that he was in this way led to some of his most remarkable results.

For details and for references to the literature the reader may consult the papers mentioned and especially the full analysis and bibliography given in the first volume of Dickson's monumental history of the theory of numbers. We shall mention here only the one or two facts which we need as affording the point of departure for some additional remarks. Let us consider the infinite sequence of integers $u_0, u_1, u_2, \cdots$ defined uniquely in terms of the initial numbers u_0,

[1] See Nörlund's Encyclopædia article.

[2] *Quarterly Journal of Mathematics*, vol. 48, No. 192, 1920.

$u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = \alpha$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. Let $r_0, r_1, r_2, \cdots$ be in order the least non-negative residues of the integers $u_0, u_1, u_2, \cdots$ with respect to the modulus m, where m is a given integer. A fundamental property of the sequence $r_0, r_1, r_2, \cdots$ is the following: The elements in the sequence, after a finite number at the beginning, repeat themselves in cycles of μ elements, where μ is a certain characteristic integer associated with the sequence. In view of this fact the sequence $u_0, u_1, u_2, \cdots$ is said to be periodic modulo m. Many important properties of the sequence are intimately connected with this property of periodicity. Accordingly a systematic study of the latter property leads to a considerable number of results of genuine interest. For these the reader is referred to the original memoirs.

In connection with these facts I shall now undertake to characterize some problems of considerable interest. It appears natural to look upon the periodicity property just mentioned as the analogue of simple periodicity in the theory of functions of a complex variable. If one conceives it so he can not fail to raise the question as to what corresponds, among recurrent sequences of integers, to the property of double periodicity in the theory of elliptic functions. That this question was in the thought of Lucas is evident from some remarks which he made in the article mentioned; but, so far as I know, no one has ever brought to explicit notice the analogue in consideration. If there is any property of sequences of integers (ordinary or algebraic) which truly corresponds to double periodicity in the theory of elliptic functions, an investigation of it has a large chance of proving of considerable interest.

More general sequences of integers than those already mentioned explicitly could be easily brought into consideration. If, in the last foregoing equation, the quantities $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are functions of x each of which is integral-valued for every integral value of x, then the equation defines an infinite sequence of integers in terms of k initial integers. [This might also be extended to the case where the integral values are replaced by algebraic integers; it seems also to be possible to extend the range of x by means of equations and systems of equations involving various algebraic units (as well as 1) among the differences of the independent variable x.] For this more general class of sequences—as well as for those mentioned in the sentence in brackets—one may consider such problems as those which have already been treated for the simpler case. So far, the problem has hardly been touched in the literature.

In connection with the linear recurrence relations which have been treated certain non-linear recurrence relations have come into consideration. Some of these appear in certain remarkable theorems giving necessary and sufficient conditions for the primality of certain classes of integers; they also come up in other connections. The purposes which they serve, where they have arisen in this incidental way, afford good grounds for believing that fruitful investigations

may be attached to certain classes of these non-linear equations. It may well be that the problem of double periodicity of sequences of integers is intimately bound up with certain classes of these non-linear equations.

Corresponding questions arise for linear q-difference equations where the multiplier q is an integer. So far as I know these have not been treated. I have not considered the problem sufficiently to have a clear judgment as to whether it may be a promising or a useful one.

5. Expansions generalizing the factorial series. In the solution of differential and difference equations functions have frequently arisen which have usually been represented in a two-fold way: the asymptotic character of one of these functions near infinity is set forth by means of a diverging power series; another form of expansion is found for it by means of which to put in evidence the analytic character of the function. Obviously, it is desirable, if possible, to have a single expansion of such character that it is capable of exhibiting the asymptotic properties of the function near infinity and of yielding at the same time a convenient and workable representation of it in the finite plane or in a significant portion of the finite plane. It is hardly to be expected that any single class of series will afford a form of such a tool best suited to all situations; but it may very well be true that certain classes of functions of great importance and very frequently recurring in practice are capable of representation in one or another sort of series all of which belong to a single type and possess a unitary theory. A few years ago it became apparent that factorial series and certain immediate generalizations of them serve just such a purpose to a remarkable degree.

But there appears to be a certain lack of flexibility of these series which restricts their range of application in such a way as to render them suited to meet only a part of the indicated need. In meditating on this matter a few years ago I was led to observe that a certain important class of functions, frequently in evidence in investigations in the difference calculus, were such as to afford the basis of definition for a large class of series of which the factorial series is but an instance, while many properties of the whole class of series are fundamentally quite as simple as the corresponding properties of the factorial series. These series are of the form

$$\Omega(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)},$$

where the coefficients c_n are independent of x and where $g(x)$ is a given function of the complex variable whose most frequently used properties are expressed by means of the asymptotic relation

$$g(x) \sim x^{P(x)} e^{Q(x)} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots\right),$$

in which $P(x)$ and $Q(x)$ are polynomials in x, $P(x)$ not being a constant in the special case in which $Q(x)$ is of degree less than 2. The general theory of the convergence of these series is very simple in its main aspects. Numerous important

special cases of them have appeared in the literature.[1] I believe that other special cases of these series will have a considerable use in the development of the theory of differential and difference equations. The fundamental theorems concerning the class of functions which are representable by means of them are yet largely to be discovered: the problem has been partially treated in several papers of my own.

Intimately related to this is the problem of the representation of fuctions by means of integrals of the form

$$J(x) = \int_0^\infty \varphi(t)g(x+t)dt,$$

where the function $g(x)$ has essentially the same properties as in the foregoing case. The convergence theory for these integrals is particularly simple and elegant, having many points of contact with that of the series $\Omega(x)$. I have given the main basic theorems in a paper in *The Transactions of the American Mathematical Society*, vol. 20 (1919): 313–322. I believe that special cases of these integrals will be found useful in much the same way as corresponding cases of the series $\Omega(x)$. Several important cases of the integrals $J(x)$ have appeared in the literature.

It seems to me that a promising field for investigation lies in certain subclasses of the series $\Omega(x)$ and integrals $J(x)$; they have an unusual flexibility for the representation of certain classes of functions— and precisely of some of those frequently arising in connection with differential and difference equations. This conviction has lain in my mind for a number of years; but pressure of other work has interfered, and promises still to interfere, with the work of following up the conviction to see explicitly in how far it is justified. Perhaps there are others who may take up the problem.

6. Expansions in conjugate functions. That the expansions treated in the foregoing sections are fundamental in the difference calculus (and elsewhere) has been shown by the effective use already made of special cases of them, particularly the factorial series and certain immediate generalizations of them. For a time I thought that they almost surely afforded the principal expansion problem of the difference calculus. So far the evidence still indicates that this is probably true. The series still appear to me to have all the importance which I attached to them from the beginning. But there is another class of series in the difference calculus which appear to me now to be of greater absolute, and hence of greater relative, importance than I had at first supposed. These are the series analogous to Fourier series and other expansions in orthogonal and biorthogonal functions in the infinitesimal calculus. Concerning them we know, up to the present, nothing but a few of their formal properties. We shall bring the address to a close by setting forth the nature of these series. Since the ordinary difference equation has been more in evidence in the address than the q-difference equation, we shall describe this expansion problem with reference to q-difference equations; but the

[1] See *Bulletin of the American Mathematical Society*, (2) 23 (1917): 407–425.

reader will remember that a similar expansion problem exists for the ordinary difference equation.

Let us consider the adjoint systems of q-difference equations

$$u_i(qx) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n, \tag{5}$$

$$v_i(x) - v_i(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})v_j(qx), \qquad i = 1, 2, \cdots, n, \tag{6}$$

where q is a constant whose absolute value is different from unity and where the φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and have a zero there. These systems of equations possess fundamental systems

$$u_{1j}(x), u_{2j}(x), \cdots, u_{nj}(x); \quad v_{1j}(x), v_{2j}(x), \cdots, v_{nj}(x)$$

of solutions each function of which is analytic at infinity, say, analytic for $|x| \geqq R$, R being an appropriately chosen positive constant; moreover, the constant term in the solution $u_{ij}(x)$, and that in the solution $v_{ij}(x)$ as well, is δ_{ij}, where δ_{ij} denotes unity or zero according as j is or is not equal to i. Any solution which is analytic at infinity is made up from the foregoing solutions by taking linear combinations of them with constant coefficients. We confine attention to those solutions of both systems which are analytic for $|x| \geqq R$.

If we multiply (5) member by member by $v_i(qx)$ and (6) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \delta\{u_i(x)v_i(x)\} = 0, \tag{7}$$

where δ denotes the operation defined by the relation $\delta f(x) \equiv f(qx) - f(x)$. Let a be a number such that $|a| \geqq R$. In (7) sum as to x from a to ∞, where x runs over the values $a, qa, q^2a, q^3a, \cdots$ or the values $a, q^{-1}a, q^{-2}a, \cdots$ according as $|q|$ is greater than or less than unity; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{8}$$

The first member of this equation is a non-singular bilinear form in the two sets of n variables each

$$u_1(\infty), u_2(\infty), \cdots, u_n(\infty), u_1(a), \cdots, u_n(a);$$
$$v_1(\infty), v_2(\infty), \cdots, v_n(\infty), v_1(a), \cdots, v_n(a).$$

This bilinear form may be written in an infinite number of ways in the normal form

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v),$$

where the $U_i(u)[V_i(v)]$ are homogeneous linear expressions (with constant coefficients) in the variables $u[v]$. Then with our systems of q-difference equations

we associate the boundary conditions

$$U_i(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{9}$$

$$V_i(v) = 0, \qquad i = n+1, \cdots, 2n. \tag{10}$$

Then relation (8) is satisfied in virtue of the boundary conditions alone.

Systems (5) and (9) define what may be called the u-problem; similarly the v-problem is defined by (6) and (10). The characteristic values λ are the same for the two problems. If $u_i^{(k)}(x)$ and $v_i^{(l)}(x)$ are solutions of the u-problem and the v-problem, respectively, the first for $\lambda = \lambda_k$ and the second for $\lambda = \lambda_l$, and if $\lambda_k \neq \lambda_l$, then we have the fundamental condition of conjugacy, namely,

$$\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(l)}(at^{s+1}) = 0, \tag{11}$$

where $t = q$ or q^{-1} according as $|q|$ is greater than or less than unity.

Let us now suppose that the problem is set up so that we have the infinitude of characteristic values $\lambda_1, \lambda_2, \lambda_3, \cdots$ and corresponding solutions of the u-problem and the v-problem; and let us suppose that the first member of (11) is different from zero when l and k are equal. Then if given functions $f_i(x)$, $i = 1, 2, \cdots, n$ have suitable expansions in the form

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the functions, these coefficients c_k have the following values

$$c_k = \frac{\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) f_i(at^s) v_j^{(k)}(at^{s+1})}{\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(k)}(at^{s+1})}, \qquad k = 1, 2, 3, \cdots.$$

This is sufficient to put in evidence the formal aspects of an expansion problem in the q-difference calculus analogous to expansions in orthogonal and biorthogonal functions in the infinitesimal calculus. Two modifications of it may be suggested. In one of these the rôle of the point infinity may be played by the point zero; that is, the summations may be from a to 0 instead of from a to ∞. This gives rise to a problem altogether analogous to the one already described; in fact, one of these problems goes essentially into the other by changing x into $1/x$. The second modification consists of a sort of combination of the two just mentioned, the summation being from 0 to infinity in the same sense in which a Laurent series affords a summation from $-\infty$ to $+\infty$. The latter form of expansion promises certain advantages over the other two.

Corresponding problems arise in connection with ordinary difference equations. Moreover, if we apply certain limiting processes which have become classic through the investigations of Volterra, we are led through to integro-difference equations and integro-q-difference equations. For these functional equations

there also exist expansion problems analogous to that which we have just treated. In each case the first formal aspects of the problem lie close to hand. It is of considerable importance to have a detailed analysis of the character of the functions which are representable in the form of certain of these expansions. No investigation of this type has yet appeared in the literature.

ON DANIEL BERNOULLI'S "MORAL EXPECTATION" AND ON A NEW CONCEPTION OF EXPECTATION.

By CHARLES JORDAN, University of Budapest.

1. Mathematical Expectation. Even those events whose accomplishment is not certain, but only more or less probable, exercise before their happening some influence over us. Therefore we attribute in advance to these events more or less utility and value.[1] From the beginning, the philosophers have agreed that this value may be considered as being proportional to the probability p of the event; moreover that this valuation depends upon the nature of the event itself. For instance, if a certain gain is expected, the greater it is, the greater the value of the expectation will be. But the philosophers soon disagreed about the degree of appreciation.

First, mathematicians and philosophers thought that the value of the expected gain might also be taken as proportional to its magnitude. If we denote the probability of the event by p, and the expected capital by x, then from this point of view the value of the expectation would be equal to $\lambda \cdot px$, where λ is an individual factor of proportionality. Since in the calculus of probability the quantity $p \cdot x$ is called mathematical expectation, the mentioned hypothesis may be thus enunciated: The effect, or utility, and hence the valuation of a coming event is proportional to its mathematical expectation; therefore our actions are regulated by the principle of mathematical expectation.

If we expect one of several events, which may respectively produce the gains $x_1, x_2, \cdots$ or x_m, then the mathematical expectation of these eventualities is the sum of the partial mathematical expectations; *i.e.*, $\Sigma p_i x_i$.

Mathematical expectation is closely connected with arithmetical mean; indeed, if we suppose that the aggregate of the n possible cases contains the quantity x_1 n_1 times, and the quantity x_2 n_2 times, and so on, and in the end the quantity x_m n_m times, then the mean value of the quantities contained in the aggregate, is $\Sigma(n_i/n)x_i$. Taking into account the fact that, according to the frequency definition of mathematical probability, n_i/n is equal to the probability p_i of the quantity x_i, we conclude that the mathematical expectation of the quantities $x_1, x_2, \cdots x_m$ is equal to the arithmetical mean of the quantities constituting the aggregate of the possible cases.

[1] The terms "utility," "value" and kindred expressions are used here without attempt at an analysis of their various possible significations in much the same manner as certain undefined terms are used in geometry. For an extended analysis of "value" see "*The Measurement of General Exchange-Value*" by C. M. Walsh, Macmillan, 1901.—EDITOR.

Accepting the principle of mathematical expectation, it follows, if p_i is the probability that our capital x increases by the quantity Δx_i, that the effect or utility ΔU_i of this change will be expressed by $\Delta U_i = \lambda p_i \Delta x_i$; and considering all the possible eventualities the total change of the *utility* will be $\Delta U = \lambda \Sigma p_i \Delta x_i$, where $\Sigma p_i = 1$.

According to this principle, we may class the events or operations under three categories: (1) favorable operations, corresponding to $\Delta U > 0$; (2) indifferent operations, corresponding to $\Delta U = 0$; (3) unfavorable operations, corresponding to $\Delta U < 0$.

If our actions were really directed by the principle of mathematical expectation, we should only participate in operations belonging to the first group. Our readiness to participate is an increasing function of ΔU.

Experience shows that this is not the case. Actually, every day we see people take part in lotteries, insurances and other transactions in which $\Delta U < 0$. On the other hand, as Nicolas Bernoulli has shown in the Petersburg paradox, there are cases in which the mathematical expectation ΔU is unlimited, and nevertheless nobody would engage in the corresponding transaction. Moreover, in every operation at least two persons are opposed to each other. If the mathematical expectation of the first is greater than zero, then that of the second must necessarily be less than zero; so that for at least one of the participants the operation must be unfavorable. According to the principle, this person should refuse to take part. No operations would be possible, except those in which $\Delta U = 0$, and they would be superfluous.

2. Moral Expectation. The principle of mathematical expectation being unacceptable, there arose the need to look for other principles. Daniel Bernoulli [1] was the first to give the following new hypothesis: If our fortune is x and p is the probability that it will increase by Δx, then the value of the expectation is, as in the former hypothesis, directly proportional to the probability p and to the expected gain Δx, but is moreover inversely proportional to the fortune x; that is

$$\Delta U = \lambda p(\Delta x)/x.$$

It results from integration that, if p is the probability that our fortune will become equal to x, the value of its utility is

$$U = \lambda p \log (x/x_0),$$

where x_0 is the initial sum or that sum which has no effect at all on us ("threshold of consciousness").

If there is a probability p_1 that our fortune will become x_1, and p_2 that it will become x_2, and so on,[2] then the value of these eventualities is, according to this principle, the sum of the values of the different expectations; *i.e.*,

$$\overline{U} = \lambda \Sigma p_i \log x_i - \lambda \log x_0 = \lambda \log [x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m}] - \lambda \log x_0.$$

[1] *Specimen Theoriæ novæ de Mensura Sortis*, 1738.

[2] We will call the aggregate of these conditions (A).

If our fortune is x, its utility or value to us (setting $p = 1$ before the risk is taken) is

$$U = \lambda \log x - \lambda \log x_0.$$

Therefore we conclude that the value of the above expectations is the same as if our fortune were actually equal to $x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}$. This last quantity is called, after D. Bernoulli, "moral fortune."

Let us return to the aggregate of the possible cases mentioned before; the geometrical mean of the quantities contained in it is

$$G(x_i) = \sqrt[n]{x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}} = x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}.$$

Hence the moral fortune (or capital) is equal to the geometrical mean of the different possible quantities.

If our capital before the transaction was x, and if, after it, we are placed in the conditions (A), then the change in the value will be

$$\overline{U} - U = \lambda \log [x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}] - \lambda \log x.$$

The quantity $G = [x_1^{p_1}x_2^{p_2}\cdots x_m^{p_m}]/x$, the relative change of the moral fortune, is called *moral expectation.*

The effect, utility and value of the change is proportional to the logarithm of the moral expectation.

From this second point of view we can also class the events under three categories—favorable, indifferent, and unfavorable events, according as $G > 1$, $G = 1$ or $G < 1$.

One of the most important consequences of D. Bernoulli's hypothesis is the principle called that of decreasing utility; *i.e.*, ΔU decreases with increasing x.

To compare the two hypotheses which we have considered, let us introduce a quantity closely connected with the mathematical expectation. If x was our capital, and if in consequence of an operation the different possibilities are those described under (A), then the mathematical expectation of the gain is $\Sigma p_i x_i - x$; let us denote by $A = \Sigma p_i x_i/x$ the *relative* mathematical expectation of our fortune. It results that if the operation is favorable from the first point of view, then $A > 1$; if it is indifferent, $A = 1$; and if it is unfavorable, $A < 1$. The quantity A is the arithmetical mean of the quantities x_1/x, x_2/x, $\cdots$ x_m/x, contained in the aggregate of possible cases. The corresponding moral expectation G is the geometrical mean of the same quantities, that is

$$G = (x_1/x)^{p_1}(x_2/x)^{p_2}(x_3/x)^{p_3}\cdots(x_m/x)^{p_m}.$$

3. The moral expectation is *always smaller* than the corresponding mathematical expectation. Indeed, the arithmetical mean is greater than the geometrical mean of the same quantities. It results that all the operations unfavorable from the first point of view are still more so from the second, and that the indifferent operations and some of those which are favorable from the first point of view are unfavorable from the second.

4. Let x be our fortune, p the probability of winning a sum equal to b, and q the probability of losing a. If $p + q = 1$, the corresponding moral expectation is:

$$G = \left[\frac{x + b}{x}\right]^p \left[\frac{x - a}{x}\right]^q . \tag{1}$$

Thus, if $a = x$, $G = 0$. For this discussion, it will be assumed that $a < x$,—that the entire fortune is not put at risk. If the operation is indifferent from the point of view of mathematical expectation, we have $bp = aq$, hence:

$$G = \left[1 + \frac{qa}{px}\right]^p \left[1 - \frac{a}{x}\right]^q .$$

To show that if a and x retain their value unchanged, G increases with increasing p, it is sufficient to demonstrate that

$$\frac{d}{dp} \log G > 0.$$

This differential quotient can be written as follows:

$$-\log\left[1 - \frac{a}{px + qa}\right] - \frac{a}{px + qa} .$$

The expansion of this logarithm in power series shows that the quantity above is greater than zero.

Therefore, the capital x and the risked sum a remaining unchanged, an operation indifferent, from the first point of view, becomes more and more unfavorable from the second, as p decreases; *i.e.*, the riskier a game is, the more unfavorable it is. This judgment coincides with everyday experience.

5. The *capital* of a gambler has no effect on the mathematical expectation of a gain, but the moral expectation of a gain increases, in unfavorable operations, if his capital becomes greater; *i.e.*, the influence of a risk is always greater for a poorer man than for a richer, conformable to ordinary opinion.

This can be proved by showing that under these circumstances $d/dx \log G > 0$. From (1) it results that

$$\frac{d}{dx} \log G = \frac{(qa - pb)(1 + x) + ab}{x(x - a)(x + b)} .$$

If $qa \gtrless bp$ (unfavorable operations), as x, a, b, p, and q are positive, the above differential quotient is certainly greater than zero.

6. From the first point of view the *loss* of a sum has the same effect, only in the opposite sense, as the gain of the same sum. From the second point of view the influence of the loss is greater than that of the gain. Indeed the effect of the gain a is proportional to $\log (1 + a/x)$, and the effect of the loss to

$\log (1 - a/x)$, but

$$\left|\log\left(1+\frac{a}{x}\right)\right| < \left|\log\left(1-\frac{a}{x}\right)\right|$$

as is easily seen by expansion of the logarithms in powers of a/x.

7. From the first point of view, the *dividing* of a risk between two chances has no effect at all; from the second, it is favorable. Indeed if q is the probability of losing, if x is our capital and if we risk on one chance the sum $2a$; then from the second point of view the effect will be proportional to $\log (1 - 2a/x)^q$. If on the other hand we risk the sum a on each of two chances, the effect will be proportional to $\log (1 - 2a/x)^{q^2}(1 - a/x)^{2pq}$; we have

$$\log\left(1-\frac{2a}{x}\right)^q < \log\left(1-\frac{2a}{x}\right)^{q^2}\left(1-\frac{a}{x}\right)^{2pq},$$

which is equivalent to $0 < a^2/x^2$. This discussion applies to such risks as may bring only loss—such as the risks of transportation, but even in risks which could augment our capital we would be led to the same conclusion.

8. Accepting the second hypothesis the *insurances* must necessarily be favorable, from both points of view, to the insurance companies, otherwise they could not be compelled to accept the risks.

The insured are in a different position; they are involved in risks which diminish their moral fortune; consequently, if they pay less than this diminution to get rid of the risk, the operation is advantageous for them from the second point of view, though unfavorable from the first.

For instance, let x be the capital of the person who insures, and b the value of the object to be insured, q the probability of its loss, a the sum paid for insurance, y the capital of the company, and $q = 1 - p$.

From what we have seen, the moral fortune of the person, if he does not insure, is $x^p(x - b)^q$; if he insures, $x - a$. The operation will be favorable to him from the second point of view, if

$$x^p(x-b)^q < x - a.$$

The moral fortune of the company is, if it does not accept the insurance, equal to y; and if it does, to $(y + a)^p(y + a - b)^q$; the operation will be favorable to it if

$$(y+a)^p(y+a-b)^q > y.$$

It is easy to show that there are values of a which satisfy at the same time both of the inequalities. For these values the tariff will be favorable to both parties, *i.e.*, the insurance will increase the moral fortune of both. This corresponds to the cases in which insurance is considered favorable in practice. Thus, while the hypothesis of mathematical expectation cannot account for insurance, moral expectation gives a good explanation of it.

It is true that an insurance contract does not produce wealth, but it increases moral capital. Its value is measured by this increment.

9. Daniel Bernoulli's hypothesis is greatly supported by Weber and Fechner's celebrated *psycho-physical law*, according to which the sensation y is proportional to the logarithm of the corresponding stimulus x; *i.e.*,

$$y = \lambda \log (x/x_0).$$

Hence this law may be regarded as an extension of Bernoulli's hypothesis to natural phenomena. It has been verified in many instances in which measured quantities are compared with their estimated magnitudes. For instance, if we compare Ptolemy's star-magnitudes, y, to the photometric intensity x of their brightness, we have

$$y_2 - y_1 = -2.5 \log (x_2/x_1).$$

Analogous formulæ can be found if we compare the estimated pitch of sound with the corresponding number of vibrations per second; or by comparing estimated values with measured values of temperature, weight, distance, height, etc.

The law can moreover be verified by the "difference thresholds," that is, by the least observable differences of the stimuli.

10. Let us now apply D. Bernoulli's law to the problem of *contributions*. Let x be the contributor's capital and a the tax; the effect of the tax will be the same to everybody except for the factor λ, if we have

$$y/\lambda = \log (x - a)/x = \text{constant},$$

that is, if the taxes are proportional to the fortunes of the contributors. On the contrary according to the hypothesis of mathematical expectation the tax, to have the same effect, should be the same for everybody whatever his fortune may be. This would be considered in practice as quite unreasonable.

11. The importance of D. Bernoulli's theory is increased by the fact that the modern *theory of value* is founded upon it. It was the starting point for Gossen's, Jevons', Walras', Edgeworth's and Pareto's works.

If a person possesses the quantity x_i of a commodity C_i, then the *utility* of this quantity of C_i is expressed by

$$U = \int_0^{x_i} f_i(t)dt,$$

where $f(t)$ is a non-increasing function of t; $f(t)$ being the degree of utility of the quantity dt and $f_i(x_i)$ the *final degree of utility*, *i.e.*, corresponding to the last element dt of x_i. This formula is a generalization of D. Bernoulli's hypothesis.

If a person possesses the quantity x_1 of a commodity C_1, and x_2 of C_2 and so on, then the total utility of his goods will be

$$U = \Sigma \int_0^{x_i} f_i(t)dt.$$

This function plays a rôle in economics analogous to that of entropy [1] in

[1] For a statement of the entropy principle see Clerk Maxwell, *Theory of Heat*, Longmans, 1908, p. 193.

physics. People act in such a manner that the total utility of their possessions increases. Production, exchange and consumption are aptly explained by it. For instance, in exchanges an equilibrium will be reached if the final degrees of utility of every commodity possessed by a person are equal.

There was but one difficulty: the utilities concerning different persons, being subjective, cannot be compared or added, so that it seemed impossible to speak of the total utility relative to a group of persons, though it would be very useful to do so.

This difficulty was removed by Edgeworth in his "*Mathematical Psychics*" (1881) where he gave a method which, by means of determining experimentally the curves and surfaces of "indifference" corresponding to a group of persons, enables us to reach the differential equations expressing the total utility of their goods.

12. New Conception of Expectation; Harmonic Expectation. Among the mathematicians there were several [1] who accepted the principle of mathematical expectation but refused that of moral expectation as being quite arbitrary, not seeing that *a priori* the two principles are equally acceptable, the first being based on the arithmetical mean of the possible cases, the second on their geometrical mean.

As all measures are inevitably affected with errors, the natural laws derived from experience can only approximately interpret physical facts. From time to time they are superseded by more general and more accurate laws. D. Bernoulli's law, therefore, though much nearer the truth than that of mathematical expectation, will nevertheless ultimately share the same fate.

Some of its imperfections can be easily pointed out; for instance, while accounting for the threshold of sensations, it asserts that there is no upper limit for them. The sensations increase, it states, indefinitely with the stimuli. But we know that this is not true; we know that there is an upper limit of sensations, and that when it is reached, by increasing the stimuli there is no change at all.

Consideration of this fact, and of others, has led me to conceive and formulate an analogous law, based on the harmonic mean of the possible quantities.

Let x be the fortune of a person, x_0 the threshold of fortune corresponding to that of consciousness; let a be a certain constant, λ a factor differing from one individual to another; Δx the increment of the fortune, and Δy the corresponding increment of sensation or utility. Let us accept the hypothesis that we have

$$\Delta y = \lambda(x_0 + a)\Delta x/(x + a)^2.$$

By integration we get the utility y produced by the fortune x:

$$y = \lambda\left[1 - \frac{x_0 + a}{x + a}\right]. \qquad (2)$$

It results that λ is the upper limit of the utility ($x = \infty$). Moreover the

[1] *e. g.* Bertrand, *Calcul des Probabilités*, p. X, p. 65.

difference of the utility of the quantities x_2 and x_1 is

$$y_2 - y_1 = \lambda(x_0 + a)\left[\frac{1}{x_1 + a} - \frac{1}{x_2 + a}\right]. \tag{3}$$

In the problems considered before, this formula gives results analogous to D. Bernoulli's hypothesis; perhaps even they are more satisfactory. For instance in the problem of contributions, if we want to tax people so that, according to this principle, the produced disutility is the same for everybody, then, if a person possesses a capital x_1, the tax $x_1 - x_2$ is obtained by putting in (3) the quantity $(y_2 - y_1)/\lambda(x_0 + a) = -1/c = \text{constant}$. It results that the tax is

$$(x_1 - x_2) = (x_1 + a)^2/(x_1 + a + c).$$

Thus this third principle leads to progressive taxes. After having paid the tax, the fortune of the contributor will be:

$$x_2 = \frac{(c - a)(x_1 + a) - ac}{x_1 + a + c}.$$

It follows that there is a maximum of x_2; indeed for $x_1 = \infty$, $\lim x_2 = c - a$.

If p_1 is the probability that our fortune will become equal to x_1, and p_2 the probability that it will become x_2, etc., then the value of these eventualities is, from our third point of view, according to (2) and to § 1 (if $\Sigma p_i = 1$) the following

$$y = \lambda\left[1 - (x_0 + a)\Sigma\frac{p_i}{x_i + a}\right] = \lambda\left[1 - (x_0 + a)\frac{1}{H}\right],$$

where H is the harmonic mean of the quantities $x_i + a$; *i.e.*, the utility y is the same as if our fortune were equal to $H - a$. If it was, before the operation, equal to x, the change of the value is

$$\Delta y = \lambda\left[\frac{1}{x + a} - (x_0 + a)\frac{1}{H}\right];$$

this quantity can be termed *harmonic expectation*.

It is hardly necessary to add, in conclusion, that before accepting this principle it would be necessary to prove that it interprets physical facts with more accuracy than Weber's law.

SIMPLE DERIVATIONS OF THE FORMULAS FOR THE DISPERSION OF A STATISTICAL SERIES.

By C. H. FORSYTH, Dartmouth College.

1. Definition of dispersion and summary of certain principles. The purpose of this paper is to offer simple derivations of certain important formulas for dispersion along lines which the writer believes to be either more general or more familiar to the average mathematician or statistician than those employed else-

where.[1] It is believed, however, that a preliminary statement in this section of those elementary principles which lead up to and enter into the derivations would be very desirable.

The dispersion σ of a set of measurements or observations $x_1, x_2, \cdots x_n$ is defined by the relation

$$\sigma = \sqrt{\frac{1}{n}\Sigma(x_r - \bar{x})^2},$$

where $\bar{x}$ denotes the arithmetic average or mean of the observations. The conception of dispersion is of the greatest importance in investigating the relative *consistency* of sets of observations because it constitutes a good measure of the way the values of the observations deviate from the mean.

The formula quoted above is only occasionally employed in practice in the form given because a set of observations is usually given in the form of a frequency distribution; that is, the observations are usually classified according to equal intervals of measurement so that to each representative observation (usually the middle of the interval) there corresponds a frequency. The successive pairs of representative observations and corresponding frequencies constitute a frequency distribution.

The most natural means of handling a frequency distribution has been found to be by the method of moments where the n-th moment of a frequency distribution can be defined by the relation

$$n\text{-th moment} = \Sigma x^n y,$$

where x denotes the values of the representative observations and y the corresponding frequencies. It is important but obvious that the value of any moment of a frequency distribution is but an approximation of the value sought.

It is usually desirable to standardize moments by dividing them by the zero-th moment or the total frequency to obtain *unit* moments (usually referred to simply by the term *moments*) and by employing as a standard vertical axis of reference the axis through the mean. In the latter case the preliminary employment of a trial mean or axis of reference affords a great saving of labor. That is, the labor of computing the moments about an arbitrary axis of reference and transforming or interpreting them in accordance with simple formulas based upon ordinary translation to give moments about the mean is far less than that found necessary in computing the latter moments directly. Several of these formulas may be stated verbally as follows:

(*a*) The first (unit) moment about any axis of reference is the arithmetic average of the deviations from the trial mean and constitutes the correction to be made to this trial mean to give the true mean;

(*b*) the second (unit) moment about a trial mean is corrected to give the second

[1] Arne Fisher, *The Mathematical Theory of Probability*, 1922, pp. 118 ff.
Coolidge, J. L., The Dispersion of Observations, *Bulletin of the American Mathematical Society*, vol. 27 (1921), p. 439.

moment about the true mean by subtracting the square of the first moment about the trial mean.

It is evident that

(*c*) the square root of the second unit moment about the mean is identically the dispersion.

Further considerations show also that

(*d*) the square of the dispersion corresponding to the sum or difference of several independent variables is equal to the sum of the squares of the dispersions of the separate variables.[1]

Let us repeat the statement that the values of the moments obtained through the processes outlined above are necessarily approximations because of the form in which the observations are given although the approximations can be made close enough for most practical purposes.

The formulas stated above are all that are required in the derivations offered in this paper and we can now approach the particular problem to be considered.

2. Two kinds of frequency distributions. Frequency distributions can be arranged into two quite different classes according to whether they can be employed to determine the value of an important ratio or probability or not. Those of the latter type are important and common. An example of that type would be a distribution of the lengths of a large number of ears of corn. However, the most that can be expected from the investigation of such a distribution is the determination of the most representative observation and an idea of the consistency of the observations taken as a whole. We shall concern ourselves here with distributions of the former type because they lead not only to the same kind of conclusions as the other type but also to a determination of an important ratio or probability and an important *diagnosis* of any inconsistencies or apparent lack of reliability.

It will be recalled that there *may* be two ways of arriving at the value of a probability: *a priori* and empirically. Analogously, if the probability of an event is known and constant, the value of the dispersion *may* be determined in each of two ways. The empirical way is the one which has already been outlined in connection with moments. If the probability of the event in a single trial is $p\ (=1-q)$, then the probabilities of a failure every time, of exactly one occurrence, of exactly two occurrences, etc., in n trials are given by the successive terms of the expansion of the binomial

$$(q+p)^n = q^n + nq^{n-1}p + \cdots + p^n. \qquad \text{(I)}$$

We can think of the terms of this expansion as frequencies of a frequency distribution and the first moment about an axis of reference taken at the "q^n" term is easily found to be np; that is, the mean $= np$. Likewise the second moment about the same axis is found to be $np + n(n-1)p^2$ and according to formulas (*b*) and (*c*) of the preceding section the square of the dispersion becomes

$$\sigma^2 = npq.$$

[1] For the derivation of this relation see Jones' *First Course in Statistics*, London, 1921, p. 158.

It will be noticed that it is assumed that the probability of the event in a single trial remains constantly equal to p throughout such an investigation. In the next sections we propose to establish formulas for the dispersion when the value of the probability of the occurrence of the event is allowed to vary in important but prescribed ways.

3. Statistical series.[1] **Definitions.** We will suppose that the occurrence or the non-occurrence of a certain event has been noted and tabulated nN times and that these observations are arranged in N sets each of n observations. If we denote the number of times the event occurs in the first set by r_1, in the second set by r_2, etc., then the frequency sequence r_1, r_2, r_3, etc., is called a *statistical series*.

We shall differentiate between three kinds of statistical series: if the probability of the event in question remains the same throughout the investigation, the series is called a *Bernoullian* series; if the probability varies within each set but alike for all sets, the series is called a *Poisson* series; if the probability remains constant throughout each set but varies from set to set, the series is called a *Lexian* series.

We shall let $p\ (=1-q)$ denote the arithmetic average of the probabilities of the event in question regardless of whether the probabilities are always the same, vary from observation to observation within a set, or vary from set to set. That is,

$$p=\frac{1}{n}(p_1+p_2+\cdots+p_n) \qquad \text{or} \qquad \frac{1}{N}(p_1+p_2+\cdots+p_N)$$

according as the series is Poisson or Lexian, where the subscripts follow the order of observations or of sets according to the type of series.

4. Derivations of the formulas for dispersion. We propose to study the most probable value of r_i or the most probable number of occurrences of the event in question in n trials and the nature of the stability of this most probable value by means of dispersion. Hence, if this most probable value differs in several investigations, the best estimate of the most probable value at our disposal in that case might well be taken to be the arithmetic average of the various values. Moreover, in one type of series the value of the dispersion will naturally vary from set to set and as we shall deal almost entirely with the squares of dispersions we shall take the liberty of employing as the most probable value of the square of the dispersion in that case the arithmetic average of the squares of the dispersion of the different sets.

Let us consider first the means of the different types of series:

The Bernoullian series is, of course, the type with which we are familiar and we can write at once that the mean of that type is

$$M_B=np.$$

[1] Lexis, *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg, 1877. The original development is due to Lexis but the final adaptation and terminology are due to Charlier.

As all the sets of a Poisson series are presumably the same or at least comparable, the mean of such a series is the mean of any set of that series or the arithmetic average of the means corresponding to the individual observations of any set, or

$$M_P = \frac{1}{n}(np_1 + np_2 + \cdots + np_n) = np.$$

As all the observations of any set of a Lexian series are comparable, the mean of a Lexian series is the arithmetic average of the means of all the sets, or

$$M_L = \frac{1}{N}(np_1 + np_2 + \cdots + np_N) = np.$$

We have then that the mean of a Poisson series or of a Lexian series is the same as the mean np of a Bernoullian series whenever p is the arithmetic average of the probabilities of that series.

Let us now consider the dispersions of the different types of series. We have already shown that the square of the dispersion of a Bernoullian series is

$$\sigma_B^2 = npq. \tag{1}$$

Moreover, we can evidently apply this formula to obtain p_iq_i as the square of the dispersion corresponding to the single i-th observation of any set of a Poisson series and if we apply formula (d) of § 1, the square of the dispersion for any and therefore for all sets of a Poisson series is $\sigma_P^2 = \Sigma p_iq_i$, where the summation is to extend from $i = 1$ to $i = n$ inclusive. There is a rather pretty way of performing this summation which will be useful again later: write $p_i = p + (p_i - p)$ and $q_i = q - (p_i - p)$. Hence,

$$p_iq_i = pq - (p - q)(p_i - p) - (p_i - p)^2,$$

whence it is easily verified that $\Sigma p_iq_i = npq - \Sigma(p_i - p)^2$ and we obtain

$$\sigma_P^2 = \sigma_B^2 - \Sigma(p_i - p)^2. \tag{2}$$

As the probability remains constant throughout any, say the i-th, set of a Lexian series, the square of the dispersion of that set of the series is np_iq_i which is about the mean np_i of that set. This is one of the rare occasions, however, when we desire the dispersion about a point other than the mean of the set, namely, the mean np of all sets, and it becomes necessary to reverse the application of formula (b) and *add* $(np_i - np)^2$ which is the square not only of the difference between the means but also, according to formula (a), of the first moment about the trial mean np. The square of the dispersion corresponding to all N sets or the whole Lexian series is then the arithmetic average of the values of $np_iq_i + (np_i - np)^2$ for all N sets, or

$$\sigma_L^2 = \frac{n}{N}\Sigma p_iq_i + \frac{n^2}{N}\Sigma(p_i - p)^2,$$

where the summation is to extend from $i = 1$ to $i = N$ inclusive. As the first summation is exactly the same kind (except for the upper limit) as that

considered in connection with the Poisson series, it is easily found that

$$\sigma_L^2 = \sigma_B^2 + \frac{n^2 - n}{N} \Sigma(p_i - p)^2. \tag{3}$$

Formulas (1), (2) and (3) are the standard formulas[1] which were to be derived and show clearly that the dispersion of a given statistical series yielding an average mean of np or probability p will be greater than $\sqrt{npq}$ if the value of the probability varies only from set to set and less than $\sqrt{npq}$ if the value of the probability varies only within each set.

5. Practical applications and limitations of formulas. The method of applying the formulas derived in this paper was stated concisely at the end of the preceding section. As illustrations, suppose that the following distributions were to give the results obtained in investigating the general death rates of three hypothetical countries each of which is assumed to include ten districts of population 1000. The numbers then are the number of deaths which are assumed to have occurred in one year in the respective districts.

(A)	(B)	(C)	$\lvert x - \bar{x}\rvert$	$\lvert x - \bar{x}\rvert^2$
17	18	24	10	100
16	17	22	8	64
11	10	20	6	36
12	11	19	5	25
13	9	18	4	16
14	12	10	4	16
15	19	9	5	25
16	16	8	6	36
14	10	6	8	64
12	18	4	10	100
140	140	140		10)482
				$\sigma^2 = 48.2$

The average number of deaths in all three countries is evidently 14 or $np = 14$ and hence $p = 0.014$ $(q = 0.986)$. Hence $\sigma_B = \sqrt{npq} = \sqrt{14(0.986)} = 3.72$ is the dispersion which we should expect if the probability p remained constant throughout the investigation. If, however, we compute the dispersions directly from the data in accordance with the definition given at the beginning of this paper and as outlined to the right of distribution (C), we obtain

$$(A) \quad \sigma = \sqrt{3.6} = 1.90 \pm 0.03,$$

$$(B) \quad \sigma = \sqrt{14.0} = 3.74 \pm 0.06,$$

$$(C) \quad \sigma = \sqrt{48.2} = 6.94 \pm 0.10,$$

where the probable errors are added to the right to show how much of a deviation

[1] Fisher's derivations (*loc. cit.*) are restricted by the assumption that the distributions of probabilities conform to the expansion (1) and Coolidge's derivations are general but the treatment in both cases is given through the use of expected or mean values.

from the value of the corresponding dispersion might well be expected due to chance alone.

These results would lead us to conclude that greater variation in the death rate occurred within each district of country (A) than from district to district; that the variation was about the same throughout country (B); and that greater variation occurred from district to district in country (C) than within each district.

It has been found from experience that most statistical series obtained in practice are Lexian and it is obvious that this type of series is the one which it is most desirable to avoid, since the value of the basic ratio or probability obtained from any set in such a case is very apt to prove far from representative of all sets.

In the derivations of the formulas considered in this paper the case in which the value of the basic probability varies both within each set and also from set to set was ignored but it will be observed at once that this is the case which occurs most frequently, if not universally, in practice and our problem consists in practice in determining where the greatest variation occurs—within the individual sets or from set to set.

Problems will rarely be found in practice which will involve the exact conditions which are assumed in the derivations of the formulas considered in this paper. The condition which it will be most difficult to find is that of equality of number of observations in each set. It is easy to set up any number of examples based on games of chance whose conditions would be ideal and which would yield satisfactory checks upon the formulas derived here but it would be a practical impossibility to find such examples in practice; for example, it would be practically impossible to find quite a number of districts all of the same size of population as assumed in the illustration given above. Extensions of the theory considered here have been made[1] to cover this probable inequality of sets and many other situations but it is the purpose of this paper to cover merely the fundamental principles of a theory which the writer considers so important and yet so unfamiliar in this country as to merit a simpler treatment than any which he has found in print.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS.

I. SOME LIMIT PROOFS IN SOLID GEOMETRY.

By W. R. LONGLEY, Yale University.

1. Introduction. The proofs of the formulas for the volume of a cone and the volume of a sphere involve limiting processes which have always been difficult

[1] Fisher, *loc. cit.*, pp. 149 ff.

for students of elementary geometry, and usually the teacher dismisses them with an informal explanation. The proofs given in this paper are new in the field of elementary geometry, although they appear to be more direct and natural than the traditional methods. They have been tried out a number of times with college freshmen and are well within the grasp of the majority. It is not unlikely that high school students, also, may find that this treatment is illuminating.

The method is that of the integral calculus and it possesses certain theoretical advantages which are important. In the first place the method is general. The same procedure is followed for any pyramid, for any cone, and for a sphere. It introduces at an early stage of mathematical training the fundamental summation idea of the integral calculus without first developing the machinery of integration. The student who will continue the study of mathematics cannot meet this idea too often or in too many different forms. The pupil who will never study the calculus gets a brief glimpse of a powerful general method which has practical applications in certain approximate formulas for computation.

The underlying concept of dividing a solid by parallel section planes is used in most of the current texts in connection with a triangular pyramid but it is quite worth while to develop this concept a little more because of its fundamental character. As applied to the sphere this concept seems much simpler than the usual one based upon spherical sectors. It is also advantageous to use the same concept in connection with the sphere that is used with the pyramid and cone.

The method involves an algebraic formula which may very properly be taken into geometry without proof. But the proof can be made by mathematical induction and here again is an opportunity for the student to get a glimpse of a powerful general method which may never come to his notice again.

Assuming that the volume of a prism and of a cylinder are known, the only theorems to which direct reference must be made are the following:

THEOREM A. *If two straight lines are cut by three or more parallel planes, the corresponding segments are proportional.*

THEOREM B. *The area of a section of a pyramid, or cone, parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid, or cone.*

THEOREM C. *The sum of the squares of the integers from* 1 *to* n *is given by*

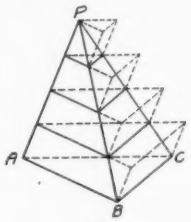
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

2. The volume of a pyramid. The figures represent a triangular pyramid, but the discussion applies to a pyramid having any polygon for a base.

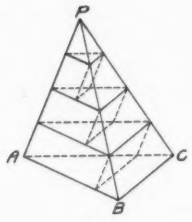
Suppose a lateral edge, AP, of a pyramid (Fig. *a*) is divided into a certain number (four in the figure) of equal parts, and that planes parallel to the base are passed through each point of division, including the vertex. These parallel planes cut out sections similar to the base and divide the altitude into the same number of equal parts as the edge AP. Suppose that a prism, with lateral edges parallel to AP and altitude equal to the distance between consecutive parallel planes, is

constructed on each section as lower base. The prisms include the pyramid completely and lie partly outside of it. These prisms form a set of circumscribed prisms.

Suppose that a prism (Fig. *b*), with lateral edges parallel to AP and altitude equal to the distance between consecutive parallel planes, is constructed on each section as upper base. These prisms lie entirely within the pyramid and form a set of inscribed prisms.



(*a*)



(*b*)

The total volume of a set of circumscribed prisms is the sum of the volumes of the separate prisms. It is obvious that the total volume of a set of circumscribed prisms is greater than the volume of the pyramid.

Suppose now that AP is divided into twice as many equal parts as before so that new section planes are passed midway between the original ones, and that a new set of circumscribed prisms is formed as before. It appears that the new set of prisms will be entirely within the old set. Hence the total volume of the new set will be less than the total volume of the old set, but will be greater than the volume of the pyramid.

It appears that, as the number of prisms is increased, the total volume of the set of circumscribed prisms will decrease, but will always be greater than the volume of the pyramid.

Similarly, it appears that, as the number of prisms is increased, the total volume of the set of inscribed prisms will increase, but will always be less than the volume of the pyramid.

Referring to the figures and counting down from the top, it is apparent that the first prism of the inscribed set is equal to the first prism of the circumscribed set. This is true also of the second prisms, third prisms, etc., until all the prisms of the inscribed set have been exhausted. There will be left only the lowest prism of the circumscribed set without any counterpart in the inscribed set. Hence for a given number of divisions of the edge AP the total volume of the set of circumscribed prisms exceeds the total volume of the set of inscribed prisms by the volume of the lowest prism of the circumscribed set. As the number of divisions of AP is increased, the altitude of each prism is decreased; hence the volume of the lowest prism of the circumscribed set is decreased, and can be made as small as we wish by making the number of divisions of AP sufficiently large. Hence the difference between the total volume of the set of circumscribed prisms

and the total volume of the set of inscribed prisms can be made as small as we wish. In other words, as the number of divisions of AP is increased, the total volume of each set of prisms approaches the same limiting value.

The preceding discussion justifies the following definition:

Definition. The volume of any pyramid is the limit approached by the total volume of a set of circumscribed (or inscribed) prisms as the number of prisms is increased indefinitely.

Theorem. The volume of a pyramid is equal to one third the product of the area of the base and the altitude.

Given the pyramid P–ABC (Fig. a). Denote the volume by V, the area of the base by b, and the altitude by h.

To prove $V = bh/3$.

Proof. 1. Suppose the edge AP to be divided into n equal parts by passing planes parallel to the base. These section planes divide the altitude into n equal parts. (Th. A.)

2. Suppose a set of circumscribed prisms to be constructed and let the prisms be numbered 1, 2, 3, $\cdots$, n from the vertex downwards. Then the altitude of each prism is h/n.

3. Let b_1 denote the area of the base of the first prism. The distance from the vertex to the base of the first prism is h/n. Hence

$$\frac{b_1}{b} = \frac{(h/n)^2}{h^2}, \qquad \text{or} \qquad b_1 = \frac{b}{n^2}. \qquad \text{(Th. } B.\text{)}$$

4. The volume V_1 of the first prism is

$$V_1 = b_1 \times \frac{h}{n} = \frac{bh}{n^3}.$$

5. The distance from the vertex to the base of the second prism is $2h/n$. Hence, with similar notation,

$$\frac{b_2}{b} = \frac{(2h/n)^2}{h^2}, \qquad \text{or} \qquad b_2 = \frac{4b}{n^2}, \qquad \text{and} \qquad V_2 = \frac{4bh}{n^3}.$$

6. Proceeding in this way,

$$V_3 = \frac{9bh}{n^3}, \qquad V_4 = \frac{16bh}{n^3}, \qquad \cdots \qquad V_n = \frac{n^2bh}{n^3}.$$

7. Let V_c denote the total volume of the set of circumscribed prisms. Then

$$V_c = \frac{bh}{n^3} + \frac{4bh}{n^3} + \frac{9bh}{n^3} + \cdots + \frac{n^2bh}{n^3} = \frac{bh}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2].$$

Hence

$$V_c = bh\left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]. \qquad \text{(Th. } C.\text{)}$$

8. As n is increased, the terms $1/2n$ and $1/6n^2$ are decreased and approach zero as a limit, and V_c approaches the value $bh/3$ as a limit.

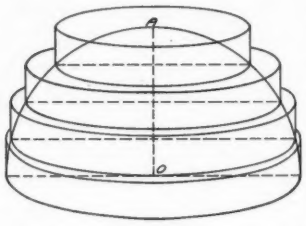
9. Hence $V = bh/3$. (Def.)

The preceding result can be obtained also by using inscribed instead of circumscribed prisms. If V_i denotes the total volume of a set of inscribed prisms obtained by dividing the altitude into n equal parts, it can be shown that

$$V_i = bh\left[\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right].$$

3. The volume of a sphere. To obtain a formula for the volume of a sphere, an argument analogous to that given for a pyramid may be used.

Consider a hemisphere and suppose the radius OA, perpendicular to the base, is divided into a certain number (four in the figure) of equal parts and that planes parallel to the base are passed through each point of division, including the point A. These parallel planes cut out circular sections. Suppose a right circular cylinder with altitude equal to the distance between consecutive parallel planes is constructed on each section as lower base. These cylinders, which include the hemisphere completely and lie partly outside of it, will be called a set of circumscribed cylinders. The total volume of a set of circumscribed cylinders is the sum of the volumes of the separate cylinders. It is obvious that the total volume of a set of circumscribed cylinders is greater than the volume of the hemisphere. Suppose now that OA is divided into twice as many equal parts as before so that new section planes are passed midway between the original ones, and that a set of circumscribed cylinders is formed as before. It appears that the new set of cylinders will lie entirely within the old set. Hence the volume of the new set will be less than the volume of the old set, but will be greater than the volume of the hemisphere. It appears that as the number of cylinders is increased, the volume of the set of circumscribed cylinders will decrease and become more and more nearly equal to the volume of the hemisphere.



As in the discussion of the volume of a pyramid, a set of inscribed cylinders may be formed by constructing cylinders on each section as upper base. The total volume of a set of inscribed cylinders is always less than the volume of the hemisphere and increases as the number of cylinders is increased. The difference between the total volumes of the set of circumscribed cylinders and the set of inscribed cylinders approaches zero as the number of cylinders is increased indefinitely, and the limiting value of the total volume of either set is the volume of the hemisphere.

The preceding discussion justifies the following definition:

Definition. The volume of a hemisphere is the limit approached by the total volume of a set of circumscribed cylinders as the number of cylinders is increased indefinitely.

Theorem. The volume of a hemisphere of radius R is $\frac{2}{3}\pi R^3$.

Given a hemisphere of radius R and volume V. To prove $V = \frac{2}{3}\pi R^3$.

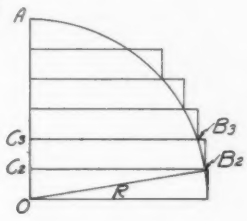
Proof. 1. Suppose the radius OA to be divided into n equal parts by passing planes parallel to the base.

2. Suppose a set of circumscribed cylinders to be constructed and let the cylinders be numbered 1, 2, 3, $\cdots$, n from the base upwards. The altitude of each cylinder is R/n.

3. The radius of the first cylinder is R and its volume V_1 is

$$V_1 = \pi R^2 \times \frac{R}{n} = \pi \frac{R^3}{n}.$$

4. Let r_2 ($= C_2B_2$) denote the radius of the second cylinder. From the figure



$$r_2^2 = R^2 - (OC_2)^2 = R^2 - \left(\frac{R}{n}\right)^2 = R^2\left(1 - \frac{1}{n^2}\right).$$

Hence the volume V_2 of the second cylinder is

$$V_2 = \pi r_2^2 \times \frac{R}{n} = \pi \frac{R^3}{n}\left(1 - \frac{1}{n^2}\right).$$

5. The radius r_3 ($= C_3B_3$) of the third cylinder is given by

$$r_3^2 = R^2 - (OC_3)^2 = R^2 - \left(\frac{2R}{n}\right)^2 = R^2\left(1 - \frac{4}{n^2}\right).$$

Hence the volume V_3 of the third cylinder is

$$V_3 = \pi r_3^2 \times \frac{R}{n} = \pi \frac{R^3}{n}\left(1 - \frac{4}{n^2}\right).$$

6. In a similar way the volume of each of the n circumscribed cylinders may be calculated.

7. Let V_c denote the total volume of the set of n circumscribed cylinders. Then

$$V_c = \pi \frac{R^3}{n} + \pi \frac{R^3}{n}\left(1 - \frac{1}{n^2}\right) + \pi \frac{R^3}{n}\left(1 - \frac{4}{n^2}\right) + \cdots + \pi \frac{R^3}{n}\left(1 - \frac{(n-1)^2}{n^2}\right).$$

Hence, by collecting terms,

$$V_c = \pi \frac{R^3}{n}\left[n - \frac{1^2 + 2^2 + \cdots + (n-1)^2}{n^2}\right].$$

8. The sum of the squares of the integers from 1 to $n - 1$ is

$$1^2 + 2^2 + \cdots + (n-1)^2 = \frac{(n-1)(n)(2n-1)}{6} = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}. \quad \text{(Th. } C.\text{)}$$

Hence

$$V_c = \pi R^3\left[1 - \frac{1}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right] = \pi R^3\left[\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right].$$

9. As n is increased, the terms $1/2n$ and $1/6n^2$ are decreased and approach zero as a limit, and V_c approaches $\frac{2}{3}\pi R^3$ as a limit. Hence the volume of the hemisphere is $\frac{2}{3}\pi R^3$. (Def.)

Corollary. The volume of a sphere of radius R is given by

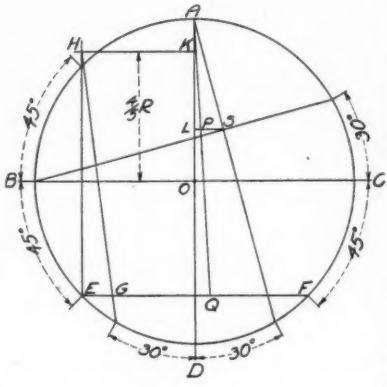
$$V = \frac{4}{3}\pi R^3.$$

II. An Approximate Construction of the Side of a Regular Inscribed Nonagon.

By T. R. Running, University of Michigan.

In the figure AP is a remarkably close approximation to the side of a regular inscribed nonagon. Taking the radius of the circle as unity, and using nine decimal places,

AP = .684040255,
Correct length of side = .684040286.



In the figure R represents the radius of the circle. The two diameters AD and BC are perpendicular. LS is drawn parallel to BC. Q bisects the line GF. AQ intersects LS in P. AP is the approximate length of a regular inscribed polygon of nine sides.

If the construction could be carried out for a circle the size of the earth's equator and AP laid off as a chord nine times, the terminal point would fall short of the initial point by about 6.3 ft., or approximately an arc which measures at the center an angle of 0″.062.

RECENT PUBLICATIONS.

Edited by D. C. Gillespie, Cornell University, to whom communications should be sent.

REVIEWS.

Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage. By L. Schlesinger. Third Edition. Berlin, Vereinigung Wissenschaftlicher Verleger, Walter de Gruyter and Co., 1922.

Among the applications of the theory of functions of a complex variable in analysis, no doubt that to the theory of differential equations is one of the most interesting. The subject was at its height about a quarter or a half of a century ago. Since then, as the author remarks in the preface to the book under review,

there has been practically a cessation of effort in the development of this theory. In the hopes of inspiring new interest in the subject and indicating some lines of research, this revision of the first edition, which appeared in 1900, has been made.

The first chapter takes up the existence theorems for differential equations in the field of real variables. The example of the motion of a simple pendulum leads to the mention of the desirability of treating differential equations in the field of the complex variable, and the remainder of the book is limited to this field. The second chapter contains a consideration of the existence of analytic solutions of differential equations of the first order, leading up to the question of singularities. Interest centers upon differential equations with fixed singularities, with consequent discussion of the Riccati equation and the linear differential equation of the first order. The third chapter treats of differential equations of the first order in which the derivative is given as an implicit algebraic function of the independent variable and again the objective is the consideration of differential equations with fixed singularities, which is carried out in detail in the succeeding chapter. There is much of interest in this latter chapter, especially in the closing sections which give a brief historical resumé of the material considered, and some indications of possible future developments.

The remainder of the book is devoted to differential equations of higher order, and is limited largely to linear differential equations and their singularities. Existence theorems are proved on the basis of the matricial idea, which is of importance in the question of generalizations, and this idea is brought in wherever feasible. The question of singularities and regular solutions leads naturally to the Fuchsian equations, and their discussion, including the Gaussian differential equation and the hypergeometric functions. There is some discussion of the nature of the solutions of differential equations in the neighborhood of non-regular points of the solutions. The concluding chapter hints at generalizations—for instance in the field of integro-differential equations—and points out some of the directions in which further research in the field considered in the book might be made.

The judging of a book ought, no doubt, to be done on the basis of objects specified in the preface and title. As an "introduction" to the subject, the book seems to the reviewer to be a little advanced. To be sure there are examples illustrating the theory as the preface would lead one to expect, but they do not appear in as great numbers as they might for an elementary understanding of the subject matter. Indications for futher research are made at two points. In view of the avowed purpose of the book, indications and suggestions for further investigation might have been given more frequently.

On the other hand, the method of approach underlying the volume, that of making the fixed singularities of the differential equations a guide for the development, seems a very happy one. Throughout the book one feels that the author has lent interest by the variety of attack on the problems considered. There is much of value to him who wants assistance in gaining a better working knowledge of the theory of analytic functions and much that will repay careful study.

T. H. HILDEBRANDT.

The Outline of Radio. By J. V. L. HOGAN. Boston, Little, Brown and Co., 1923. 256 pages. Price $2.00.

The striking achievements in the field of radio transmission of intelligence and the simplicity of the apparatus which will enable one to receive any message which may be passing have resulted in an enormous general interest in the nature of the processes which are involved. Mr. Hogan's little book aims to bring an understanding of the nature of radio processes to the average intelligent reader, whether or not he has been trained to the understanding of the physical laws of natural phenomena. For those who have no scientific training it will have the appeal of an easy style and a beginning, under each topic, with facts and laws which are matters of daily experience. Nor will they find it difficult to follow the author's explanations of the less familiar natural phenomena involved in the somewhat extended chain of transformations going to make up the complete radio message. Each of the links in this chain is taken up in its turn, and its nature explained by analogy and comparison with other more familiar phenomena. So far back does the author go in some cases in his purpose to enable the uninitiated to pick up the thread, that he occasionally approaches the metaphysical, as for example, in his comments on the nature of intelligence and that of the ether, with a result, however, altogether helpful to his purpose.

To the scientist in other fields the book also makes its appeal by reason of its complete assembly of all the elements of the radio chain, and particularly by its presentation of the numerical limits of the various quantities involved, and their peculiarities within the different ranges. The discussion of frequency and wave length is especially happy, taking the reader over the entire range of the values of each to be met in nature, from ultra-violet light at one end to the shortest sound waves on the other.

The easy style enlivens the historical review, as well as all that follows it, and a well-chosen series of diagrams also does much to help the unaccustomed reader on his way to an understanding of the nature of radio. Mr. Hogan reveals himself a good physicist and a good teacher. His book should prove a popular response to the present widespread demand for general knowledge of the nature and possibilities of radio.

J. B. WHITEHEAD.

Ausgleichsrechnung nach der Methode der kleinsten Quadrate. By V. HAPPACH. Leipzig and Berlin, B. G. Teubner, 1923. 8vo. 76 pages. Price $0.38.

This short treatise, Band 18 of Teubner's "*Technische Leitfäden*" written by "Ingenieur Vollrat Happach," gives the formulas needed for the practical use of the method of least squares. No attempt is made to explain the theory of probability, upon which they are grounded; comparatively few proofs are given. It is shown, however, that the least square hypothesis leads to the arithmetic mean and to the weighted mean of the measurements.

Most of the errors or misprints noticed appear in Sections 7 and 8; but some of these, like the occurrence of 2 as a subscript instead of as an exponent, in line 5 of page 23, would cause no trouble.

Problems are not assigned. However, sixty-three illustrative examples are worked out in considerable detail, showing forms for computation, and this is, indeed, one of the most attractive features of the book. The reader should get a good working knowledge of least squares, with its many applications.

E. L. DODD.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

ANNALS OF MATHEMATICS, second series, volume 24, no. 2, December, 1922: "Spherical representation of conjugate systems and asymptotic lines" by W. C. Graustein, 89–98; "On a short method of least squares" by B. H. Camp, 99–108; "On the convergence of the Sturm-Liouville series" by J. L. Walsh, 109–120; "The functional equation $g(x^2) = 2ax + [g(x)]^2$" by J. H. M. Wedderburn, 121–140; "On convergence factors in triple series and the triple Fourier's series" by B. M. Eversull, 141–166; "On the minimizing of a class of definite integrals" by P. R. Rider, 167–174; "A Pythagorean functional equation" by E. Hille, 175–180—No. 3, March, 1923: "On the potential of a homogeneous spherical cap, of a magnetic shell, and of a steady circular current" by C. De Jans, 181–208; "On cyclic harmonic curves" by H. Hilton, 209–212; "Multiple integrals in n-space" by P. Franklin, 213–226; "On symmetric forms in n variables" by A. Dresden, 227–236; "Algebraic fields" by J. H. M. Wedderburn, 237–264; "A theorem concerning certain unit matrices with integer elements" by H. R. Brahana, 265–270.

JOURNAL OF WASHINGTON ACADEMY OF SCIENCE, volume 13, no. 10, May 19, 1923: "The reduction of all physical dimensions to those of space and time" by A. P. Mathews, 195–210.

MATHEMATICS TEACHER, volume 16, no. 5, May, 1923: "Empirical theorems in Diophantine analysis" by R. D. Carmichael, 257–265; "Pennsylvania State course of study in mathematics" by J. A. Foberg, 266–273; "Fate and Freedom" by A. Korzybski, 274–290; "Mechanics" by G. R. Mirick, 291–294; "Varieties of minus signs" by F. Cajori, 295–301; "Correlation of mathematical subjects develops mathematical power" by C. A. Stone, 302–310.

PHILOSOPHICAL MAGAZINE, volume 46, no. 276, December, 1923: "Electronic conduction in metals" by A. Bramley, 1053–1073.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 1, January, 1924: "Stereoscopic harmonic curves" by W. F. Rigge, 29–35; "A thread of mathematical history" by Z. Ferguson, 37–44.

SCIENCE, volume 59, no. 1514, January 4, 1924: "American mathematics during three quarters of a century" by G. A. Miller, 1–6. January 11, 1924: "Mathematics and geophysics" by W. D. Lambert, 30–31.

SCIENTIFIC AMERICAN, volume 130, no. 2, February, 1924: "The Unity of Mathematics," editorial, 84.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, voume 25, no. 2, April, 1923: "Developments associated with a boundary problem not linear in the parameter" by R. E. Langer, 155–172; "Invariant points of a surface transformation of given class" by J. W. Alexander, 173–184; "Applications of analysis to the arithmetic of higher forms" by E. T. Bell, 185–189; "On the second derivatives of an extremal-integral with an application to a problem with variable end points" (Supplementary paper) by A. Dresden, 190–192; "Abstract group definitions and applications" by W. E. Edington, 193–210; "On the integrals of elementary functions" by J. F. Ritt, 211–222; "Invariants of the linear group modulo P^k" by M. M. Feldstein, 223–239; "On certain polar curves with their application to the location of the roots of the derivatives of a rational function" by B. Z. Linfield, 239–258; "Orthogonal systems of hypersurfaces in a general Riemann space" by L. P. Eisenhart, 259–280; "Ruled surfaces with generators in one-to-one correspondence" by E. P. Lane, 281–296; "Symmetric tensors of the second order whose first covariant derivatives are zero" by L. P. Eisenhart, 297–301.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3070. Proposed by J. L. RILEY, Tarleton Station, Texas.

A triangle circumscribes the circle $x^2 + y^2 = r^2$ and two of its vertices lie on the circle $(x - a)^2 + y^2 = R^2$; find the condition necessary that the third vertex may also lie on this circle.

3071. Proposed by E. L. POST, Cornell University.

Prove that, when $|h| < 1/\epsilon$,

$$1 + \frac{(x+h)}{1!} + \frac{(x+2h)^2}{2!} + \cdots + \frac{(x+nh)^n}{n!} + \cdots = c\epsilon^{mx},$$

where both c and m are independent of x, and the latter satisfies the equation $\epsilon^{mh} = m$. Note: Under the given conditions both members of the first equation considered as functions of x directly satisfy the mixed difference equation $f'(x) = f(x + h)$. The identification of these two equations is therefore of some interest.

3072. Proposed by W. H. RASCHE, Virginia Polytechnic Institute.

If the four vertices, A, B, C, and D (taken in cyclic order), of a simple quadrangle are four points in a plane rigid body undergoing complanar motion, and if two opposite sides, as AB and DC, represent vectorially the accelerations of A and D respectively, then Clifford's point of the quadrangle is the center of acceleration of the rigid body.

Note: The Clifford point associated with four complanar straight lines is the point common to the circumcircles of the four triangles formed by omitting the lines in turn.

3073. Proposed by P. E. HEMKE, U. S. Naval Academy.

Evaluate the integral:

$$\int_{\omega}^{\omega'} \log\left[\frac{(pu - pu_1)}{(pu - pv)}\right] \frac{(pu - pu_1)(pu - pu_2)du}{(pu - pv)^2},$$

where pu is here the Weierstrassian function, ω, ω' and $\omega'' = \omega + \omega'$ (ω real and ω' pure imaginary) are its half periods in the usual notation, u_1, u_2, and v are known and are represented in the Argand diagram by points on the segments $\omega\omega''$, $O\omega'$, $O\omega$, respectively.

3074. Proposed by J. H. MURPHY, Pittsburgh, Pa.

Two circles of radii, R_1, R_2, intersect. A third circle passes through their points of intersection. Find the radius of this third circle, when the sum of the areas of the two crescents cut from it by the other two circles is a maximum.

3075. Proposed by E. T. BELL, University of Washington.

What becomes of Achilles and the tortoise in a time and space which are both discrete (quantised space-time)?

SOLUTIONS.

3000 [1923, 41]. Proposed by J. ROSENBAUM, Milford, Conn.

With use of compasses alone, to construct a circle whose area shall be n times the area of a given circle, where n is any positive integer.

SOLUTION BY H. HALPERIN, Agri. & Mech. College of Texas.

Let r and R represent the radii of the given and the required circles, respectively. Then $\pi R^2 = n\pi r^2$, and $R = \sqrt{n}r$. First of all, we know how to construct, with compasses alone, the vertices of an equilateral triangle of side r. This being the case, we construct the vertices of such a triangle $A_1A_2B_1$, then the third vertex B_2 of a second equilateral triangle $A_2B_1B_2$, then the third vertex A_3 of a third equilateral triangle $A_2B_2A_3$, and so on. In this way we obtain as many points $A_1, A_2, A_3, \cdots, A_n$ as we please, which are easily seen to lie on a straight line such that $A_1A_n = (n-1)r$.

If now n is even we take $n-1$ of the A points $A_1, A_2, \cdots, A_{n-1}$; and with A_1 and A_{n-1} as centers and radii $A_1A_{(n/2)+1} = A_{n-1}A_{(n/2)-1}$ we describe arcs of circles intersecting at M, which must be on the perpendicular to A_1A_{n-1} at $A_{n/2}$. Then from the similarity of the triangles $A_1A_{(n/2)+1}M$ and $MA_{(n/2)-1}A_{(n/2)+1}$ we have

$$MA_{(n/2)+1} = \sqrt{A_1A_{(n/2)+1} \times A_{(n/2)-1}A_{(n/2)+1}} = \sqrt{(n/2)r \times 2r} = \sqrt{n}r;$$

and hence $MA_{(n/2)+1}$ is the required radius.

If n is odd we take n points, and from A_1 and A_n as centers we describe arcs with radii $A_1A_{(n+3)/2} = A_nA_{(n-1)/2}$, meeting at M which lies on the perpendicular to A_1A_n at $A_{(n+1)/2}$. We have then $MA_{(n+1)/2} = \sqrt{n}r$, the radius of the required circle.

Also solved by MICHAEL GOLDBERG, ABIGAIL E. JOHNSON, L. E. LUNN, A. PELLETIER, and the PROPOSER.

3002 [1923, 41]. Proposed by C. N. MILLS, State Normal, Aberdeen, S. Dak.

The diagonals of any quadrilateral are in length a and b respectively and are inclined at an angle A. Show that the greatest rectangle which can be drawn with its four sides passing through the four corners of the quadrilateral is $\frac{1}{2}ab(1 + \sin A)$.

SOLUTION BY R. M. MATHEWS, Wesleyan University.

Through the ends of a draw parallels inclined at an angle B to it. Then parallels through the ends of b perpendicular to the first pair make a rectangle which we may say is circumscribed, in a general sense, to the quadrilateral. The altitude of the rectangle is $a \sin B$, and the base is $b \cos (B - A)$, by a proper choice of B. Therefore, the area is

$$\Delta = ab \sin B \cos (B - A) = \frac{ab}{2}[\sin (2B - A) + \sin A].$$

The maximum Δ is obviously given by setting $B = (90° + A)/2$; we thus obtain for this maximum

$$\Delta = \tfrac{1}{2}ab(1 + \sin A).$$

It is interesting to remark that the positions of the vertices of the quadrilateral on the diagonals have no effect on the directions of the sides of the maximum rectangle. These directions are found from the angle A by bisecting the angle found by adding 90° to A; this bisector and its perpendicular are the required directions.

Also solved by MICHAEL GOLDBERG, A. M. HARDING, H. HALPERIN, ABIGAIL E. JOHNSON, E. J. OGLESBY, A. PELLETIER, A. V. RICHARDSON, J. B. REYNOLDS, and HAZEL E. SCHOONMAKER.

3003 [1923, 41]. Proposed by R. M. MATHEWS, Wesleyan University.

The angle PAM rotates around A and meets a line rotating around B in P and M. When M moves along the perpendicular bisector of AB, the locus of P is an equilateral hyperbola of which the mid-point of AB is the center. Generalize.

SOLUTION BY THE PROPOSER.

The pencil at B described by BM is perspective with the pencil at A described by AM, and the latter is congruent to the pencil described by AP. Thus (BM) and (AP) are projective pencils and the locus of P is a conic.

When BA bisects angle MAP, then BM is parallel to AP; and this happens again when BM is advanced a right angle from that position. Thus the conic has two points at infinity in mutually orthogonal directions and must be an equilateral hyperbola.

When AM passes through B, AP is the corresponding ray to BA and hence is the tangent to the conic at A. When AP passes through B, its corresponding ray is the tangent to the conic at B, and it is easily seen that it must be parallel to the tangent at A. Hence AB is a diameter and its mid-point is the center.

A generalization may be made by replacing the straight line upon which M moves by a conic through A and B. We obtain in this way four conics through A and B, the loci of M, N, P, Q, respectively. See [*1923*, 78] III. *Discussion of 2903* where a similar proof of the first part is given.

This construction is a special case of Newton's method of generating conics: two constant angles A and B in the same plane rotate around A and B while the intersection M of two sides traverses a fixed line; then the intersection P of the other two sides will in general describe a conic through A and B.

It does not seem to have been remarked that the loci of N and Q, the other two intersections of the sides of the angles, are conics through A and B. The relations of the three conics P, N, and Q may be of interest.

Also solved by H. HALPERIN, WILLIAM HOOVER, A. PELLETIER, J. B. REYNOLDS, A. V. RICHARDSON, and M. YOUNG.

3010 [1923, 76]. Proposed by F. D. MURNAGHAN, John Hopkins University.

Find an expression for the volume of the pedal tetrahedron, with respect to the tetrahedron of reference, of a point whose perpendicular distances from the sides are (x_1, x_2, x_3, x_4); from this expression show that the locus of points the feet of whose perpendiculars on the faces of the tetrahedron of reference are coplanar is Steiner's cubic surface

$$\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} = 0,$$

where the A's are the areas of the faces of the tetrahedron of reference.

SOLUTION BY THEODORE BENNETT, University of Illinois.

In Cartesian coördinates, let the faces of a tetrahedron be

$$p_i = a_i x + b_i y + c_i z + d_i = 0, \qquad i = 1, 2, 3, 4.$$

Let Δ be the determinant of this system of equations, and let α_i be the cofactor of a_i, β_i that of b_i, etc. If P_i be the vertex opposite the face p_i, we find that $P_i = \left(\frac{\alpha_i}{\delta_i}, \frac{\beta_i}{\delta_i}, \frac{\gamma_i}{\delta_i}\right)$; hence the volume of the tetrahedron is

$$V = \frac{\Delta^3}{6\delta_1\delta_2\delta_3\delta_4}.$$

If h_i be the altitude from the vertex P_i, we find that $h_i = \Delta/\delta_i r_i$, where $r_i = \sqrt{a_i^2 + b_i^2 + c_i^2}$. Knowing the volume, and the altitude from each vertex, we find the area of the face opposite P_i to be

$$A_i = \frac{\delta_i r_i \Delta^2}{2\delta_1\delta_2\delta_3\delta_4}.$$

In order to avoid any ambiguity of sign, we observe that the equations p_i can be written in such a manner that Δ is positive, and P_i on the positive side of p_i. Then h_i is positive, from the method of its formation, and hence δ_i is positive. Therefore V and A_i are also positive.

Now let $Q = (x', y', z')$ be a point whose distance from p_i is x_i, and let Q_i be the projection of Q upon p_i. The coördinates of Q_i are the values of x, y, z, obtained by solving the equations

$$\frac{x - x'}{\frac{a_i}{r_i}} = \frac{y - y'}{\frac{b_i}{r_i}} = \frac{z - z'}{\frac{c_i}{r_i}} = -x_i.$$

Hence,

$$Q_i = \left(x' - \frac{a_i x_i}{r_i},\quad y' - \frac{b_i x_i}{r_i},\quad z' - \frac{c_i x_i}{r_i} \right).$$

The volume of the tetrahedron formed by the points Q_i may be written as a determinant, from which we can eliminate x', y', z' at once, and write the result in the form

$$V' = -\tfrac{1}{6} \frac{x_1 x_2 x_3 x_4}{r_1 r_2 r_3 r_4} \begin{vmatrix} a_1 & b_1 & c_1 & r_1/x_1 \\ a_2 & b_2 & c_2 & r_2/x_2 \\ a_3 & b_3 & c_3 & r_3/x_3 \\ a_4 & b_4 & c_4 & r_4/x_4 \end{vmatrix}.$$

Expanding on the fourth column, and replacing $\delta_i r_i$ by $2A_i \delta_1 \delta_2 \delta_3 \delta_4 \div \Delta^2$, this becomes

$$V' = -\tfrac{1}{3} \frac{\delta_1 \delta_2 \delta_3 \delta_4 x_1 x_2 x_3 x_4}{r_1 r_2 r_3 r_4 \Delta^2} \left(\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} \right).$$

Using the values of h_i and V as given before, we can eliminate r_i and δ_i from V', with the result that

$$V' = -\frac{h_1 h_2 h_3 h_4 x_1 x_2 x_3 x_4}{108 V^2} \left(\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} \right),$$

which gives V' in terms of the various dimensions of the tetrahedron of reference.

If the points Q_i are coplanar, $V' = 0$, whence Q must be on the surface

$$\frac{A_1}{x_1} + \frac{A_2}{x_2} + \frac{A_3}{x_3} + \frac{A_4}{x_4} = 0.$$

Also solved by W. J. MARTIN, and A. PELLETIER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

The program of the winter meeting of the Association of Teachers of Mathematics in Southern Massachusetts, held February 9, 1924, in Fall River, included the following papers: "The imaginary in geometry," by Professor W. C. GRAUSTEIN, of Harvard University; "Standardized tests in plane geometry," by Miss VERA SANFORD, of the Lincoln School of Columbia University, and "The course in freshman mathematics at Brown," by Professor R. W. BURGESS of Brown University.

At the request of the Commission on New Types of Examination of the College Entrance Examination Board, Professor L. P. EISENHART of Princeton University has formed a committee of mathematicians to examine critically certain statistical methods used in the investigations of the Commission. The other members of the committee are Professors R. W. BURGESS, W. L. CRUM, E. V. HUNTINGTON, H. H. MITCHELL, H. L. RIETZ, and J. H. M. WEDDERBURN.

Mr. P. L. EVANS, formerly instructor of mathematics in the Manhattan High School, has been appointed instructor of mathematics and engineering drawing at Baker University.

At the University of Chicago, Associate Professor A. C. LUNN has been promoted to a full professorship of mathematics. Dr. MAYME I. LOGSDON, instructor of mathematics, has recently been appointed to a deanship in the College of Science.

Dr. PAULINE SPERRY, instructor in mathematics at the University of California, has been promoted to an assistant professorship.

Dr. GANESH PRASAD, dean of the Faculty of Science in the Benares University, has been appointed Hardinge professor of higher mathematics in the Calcutta University, in succession to Professor C. E. CULLIS.

At Cornell University, Assistant Professor ARTHUR RANUM has been promoted to a full professorship. Professor F. W. OWENS has been granted leave of absence for the second semester of the year 1923–24, and will spend the time in Europe. Dr. E. L. POST, formerly of Columbia University, has been appointed instructor of mathematics for the second semester of the current year.

Mr. H. A. SIMMONS, instructor in mathematics in the University of Michigan, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Associate Professor H. H. DALAKER, of the University of Minnesota, has been promoted to a full professorship of mathematics.

Professors G. D. BIRKHOFF, of Harvard University, and R. C. ARCHIBALD, of Brown University, will lecture at the University of California in the summer of 1924.

Professor H. F. BLICHFELDT, of Stanford University, will lecture at Columbia University during the summer session.

Professor EARL CHURCH, of the Pennsylvania Military College, has resigned to take charge of the computation and least squares adjustment of the geodetic survey of the Hawaiian Islands.

Professor JOSEPH LIPKA, of the Massachusetts Institute of Technology, died January 15, 1924, at the age of forty. Professor Lipka, who secured his doctorate from Columbia in 1912, had been on the mathematical staff of the institute since 1908, with the rank of assistant professor since 1917.

Professor JAMES HARKNESS, of McGill University, acting dean of the faculty of arts of that university, died December 7, 1923, at the age of fifty-nine years. Professor Harkness had been a member of the American Mathematical Society since 1891, and had held office as vice-president.

At Harvard University, the following have been appointed to part time instructorships of mathematics: Messrs. H. L. GARABEDIAN, E. B. HAM, E. R. C. MILES, F. W. PERKINS, T. L. SMITH, and L. E. WARD. Professor W. P. RUSSELL, of Pomona College, is at Harvard as visiting lecturer for the second semester of the current year.

Dr. PHILIP FRANKLIN, now Benjamin Peirce teaching fellow at Harvard University, has been appointed instructor of mathematics at the Massachusetts Institute of Technology, 1924–1925.

Professor R. E. WILSON of Northwestern University died January 30, 1923, at the age of fifty-one. He had recently been appointed Dean of Men. Professor Wilson was a charter member of the Mathematical Association of America.

CORRECTION—The International Congress of Mathematicians is to meet in Toronto, Canada, August 11–16, 1924, and not on the dates previously announced (*1923*, 458).

The following reports of Summer Sessions to be held in 1924 have been received.

University of California, Intersession, May 12 to June 21, and Summer Session, June 23 to Aug. 2. In addition to the usual courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses will be offered: *Intersession*: By Professor T. M. PUTNAM: Mathematical theory of investment; Solid analytic geometry; Theory of infinite series. By Professor F. IRWIN: Selected topics in the theory of equations. By Professor M. H. MCDONALD: Differential equations. *Summer Session:* By Professor G. D. BIRKHOFF: Fourier's series and their application; Mechanics. By Professor R. C. ARCHIBALD: Elementary geometry for advanced students; History of elementary mathematics. By Professor M. W. HASKELL: Higher geometry; Functions of the complex variable. By Professor D. N. LEHMER: Advanced analytic geometry; Solid analytic geometry; Theory of infinite series.

University of Chicago, first term, June 16 to July 23; second term, July 24 to August 29. In addition to the usual courses in College algebra, Plane analytic geometry, and Calculus, the following advanced courses are announced: By Professor G. A. BLISS: Functions of a real variable; Thesis work in analysis. By Professor L. E. DICKSON: Theory of Numbers, I; Thesis work in number theory. By Professor H. E. SLAUGHT: Elliptic integrals; Differential equations. By Professor M. FRÉCHET: Theory of abstract sets; Theory of probability. By Professor E. T. BELL: General theory of numbers; Theory of equations. By Professor F. R. MOULTON: Functions of infinitely many variables; Analytic mechanics, II. By Professor E. P. LANE: Synthetic projective geometry. By Doctor MAYME I. LOGSDON, Introduction to higher algebra.

Cornell University, July 5 to August 15. By Professor W. L. G. WILLIAMS: Analysis. By Professor C. F. CRAIG: Projective geometry. The following reading and research courses are also offered: By Professor J. I. HUTCHINSON and Professor CRAIG: Functions of a complex variable. By Professor VIRGIL SNYDER: Algebraic geometry. By Professor F. R. SHARPE: Hydrodynamics and elasticity. By Professors D. C. GILLESPIE and W. H. HURWITZ: Analysis. By Professors W. B. CARVER and F. W. OWENS: Projective geometry. By Professor WILLIAMS: Algebraic invariants.

Harvard University, July 7 to August 16. Elementary courses are offered in Trigonometry, Analytic geometry, and Calculus.

University of Illinois, June 16–August 9. In addition to the usual courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Doctor H. A. BENDER: Advanced algebra. By Dr. C. C. CAMP: Differential equations. By Dr. E. E. LIBMAN: Vector calculus. By Professor E. J. TOWNSEND: Theory of integration. By Professor A. B. COBLE: Differential geometry.

University of Iowa, first term, June 5–July 18. In addition to courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following courses are offered: By Dr. ROSCOE WOODS: Theory of equations; Advanced coör-

dinate geometry. By Dr. J. O. OSBORN: Differential equations. By Dr. F. M. WEIDA: Elements of statistics. By Professor E. W. CHITTENDEN: Theory of functions. Second term, July 21–August 22. By Professor J. F. REILLY: Differential equations; Green's theorem with applications. By Dr. J. O. OSBORN: Projective geometry.

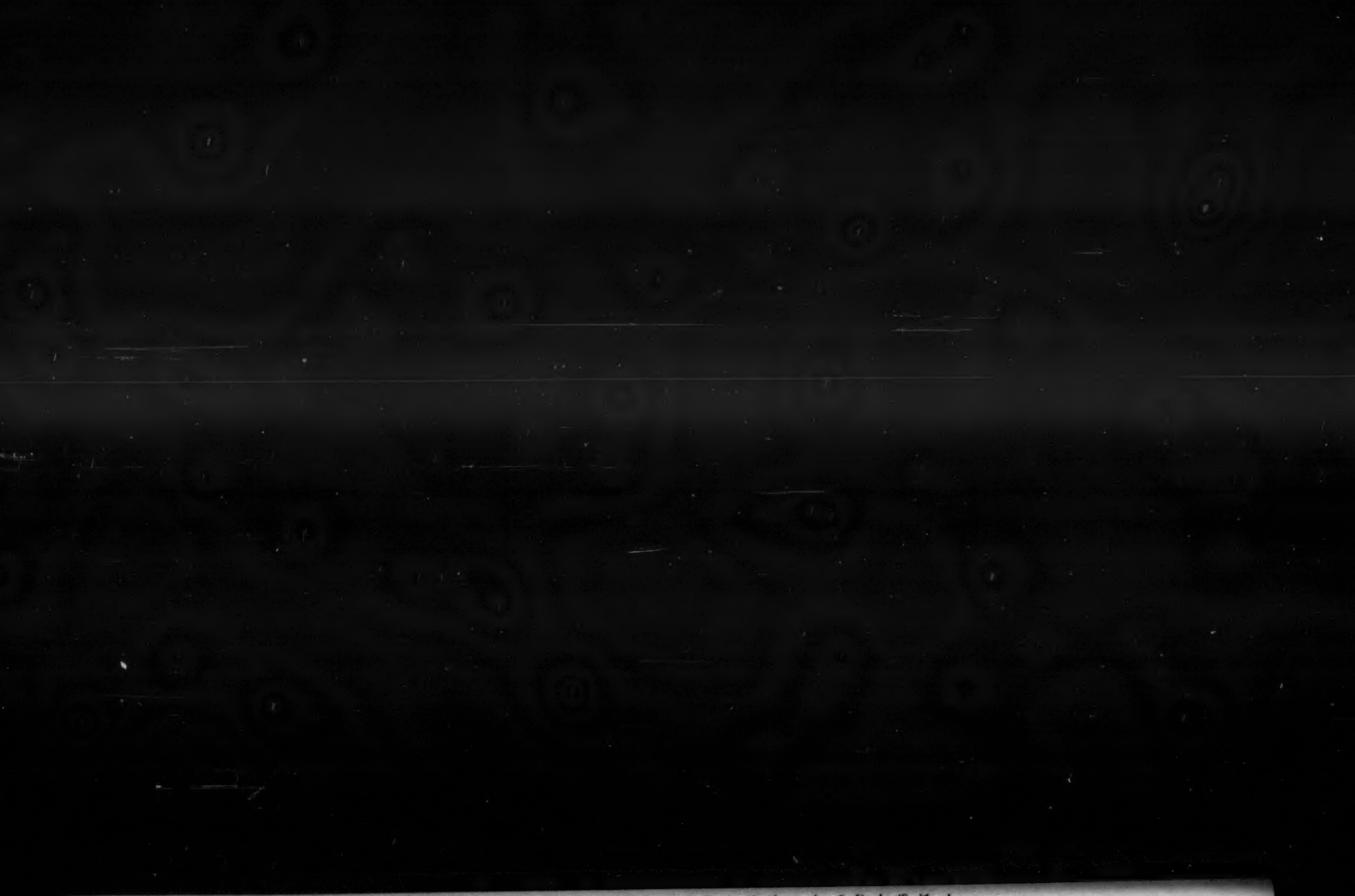
University of Michigan, June 23–August 15. By Professor W. B. FORD: Theory of functions of a complex variable; Advanced calculus. By Professor L. C. KARPINSKI: Teaching of algebra; History of mathematics. By Professor PETER FIELD: Analytic mechanics. By Professor T. R. RUNNING: Graphical methods. By Professor T. H. HILDEBRANDT: Theory of functions of a real variable. By Professor H. C. CARVER: Advanced mathematical theory of statistics; Finite differences. By Professor C. J. COE: Differential equations. By Professor NORMAN ANNING: Solid analytic geometry. By Mr. S. E. FIELD: Projective geometry. By Mr. W. A. JENKINS: Theory of probability.

University of Minnesota, first term, June 21–July 31; second term, August 1–September 5. The department of mathematics will offer an intensive course entitled: Selected topics in advanced mathematics. The topics for 1924 are: First term: By Professor DUNHAM JACKSON: Vector analysis. By Professor A. L. UNDERHILL: Differential equations. By Professor R. W. BRINK: Interpolation. Second term: By Professor W. L. HART (topic to be announced later).

University of Oklahoma, June 4 to July 29. By Professor S. W. REAVES: Differential geometry; Theory of equations. By Professor J. O. HASSLER: Analytic mechanics; Teachers course in mathematics.

University of Pennsylvania, July 7–August 16. In addition to the usual courses in Solid geometry, Trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered: By Professor G. H. HAZLETT: Introduction to functions of a complex variable. By Professor H. H. MITCHELL: Theory of probability. By Professor J. R. KLINE: Elementary statistics; Point set theory.

Columbia University, July 7 to August 15. In addition to courses in Logarithms and Trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered: By Professor H. F. BLICHFELDT: Elementary exposition of selected topics in modern mathematics; Theory of groups of finite order. By Professor W. B. FITE: Theory of functions of a complex variable. By Professor J. F. RITT: Differential equations. By Professor G. A. PFEIFFER: Projective geometry.



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